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Research Article

ON SOME BETA-FRACTIONAL INTEGRAL INEQUALITIES

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ABSTRACT

In this paper, we obtain some new integral inequalities using beta-fractional integrals in the case of two synchronous functions. For this purpose we state and prove several theorems. Our results are pioneer for the literature of integral inequalities in beta-fractional integral sense.
Keywords: Integral inequalities, beta-fractional integral.

1. INTRODUCTION

Integral inequality based on fractional derivatives is a rising trend in mathematics. There are several different fractional integral and derivative definitions. In 2014 [1], authors proposed a relatively new fractional derivative definition so called beta-derivative which is the modified version of $\propto$-derivative defined by [2]. Some articles have been focused on analytical or numerical solutions of the fractional differential equations involving beta-fractional derivative [3, 4, 5, 6, 7, 8]. Also interested reader can obtain more information on fractional integral inequalities from recent articles [9, 10, 11]. Next, we give the definition of beta-fractional derivative.

Definition 1.1. [3] Let f be a function, such that, $f:[\alpha,\infty)\to\mathbb{R}$. Then the beta-derivative is defined as:

$$D_x^{\beta}\big(f(x)\big) = \lim_{\epsilon\to 0} \frac{f\left(x+\epsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(x)}{\epsilon}$$

for all $x \geq \alpha, \beta \in (0,1]$. Then if the limit of the above exists, f is says to be beta-differentiable.

Some useful properties of this definition [3] are as follows: Assuming that $g \neq 0$ and f are two functions β-differentiable with $\beta \in (0,1]$ then, the following relations can be satisfied

$$ {}_0^{A}\mathcal{D}_t^{\beta}\big(a\,f(t)\big) + bg(t)\big) = \alpha\, {}_0^{A}\mathcal{D}_t^{\beta}\big(f(t)\big) + b\, {}_0^{A}\mathcal{D}_t^{\beta}\big(g(t)\big), \tag{1.1}$$

for all α and b are real numbers.

$$ {}_0^{A}\mathcal{D}_t^{\beta}(c) = 0, \tag{1.2}$$

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for c any given constant.

$${}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}\big(f(\mathrm{t})+g(\mathrm{t})\big)=g(t){}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}\big(f(t)\big)+f(t){}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}\big(g(t)\big). \tag{1.3}$$

$${}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}\left(\frac{f(t)}{g(t)}\right)=\frac{g(t){}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}(f(t))-f(t){}_0^{\mathrm{A}}\mathcal{D}_t^{\beta}(g(t))}{g^2(t)}\ . \tag{1.4}$$

Now, we can give the definition of beta-fractional integral.

Definition 1.2. [3] Let $f:[\alpha,b]\to\mathbb{R}$ be a continuous function on the opened interval (a,b), then the beta-integral of f is given as:

$${}_0^{A}I_t^{\beta}\big(f(t)\big)=\int_0^t\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ f(x)dx.$$

for all $x\le a,\beta\in(0,1]$. Then if the limit of the above exists, f is says to be beta-differentiable.

2. MAIN RESULTS

In this section we present our results using β-fractional integrals.

Theorem 2.1. *Let f and g be two synchronous functions on* $[0,\infty)$. *Then the following inequality holds*

$${}_0^{A}I_t^{\beta}(fg)(t)\ge\frac{1}{{}_0^{A}I_t^{\beta}\ (1)}{}_0^{A}I_t^{\beta}(f)(t){}_0^{A}I_t^{\beta}(g)(t) \tag{2.1}$$

for all $t\ge 0,\beta>0$.

Proof Since f and g are two synchronous functions on $[0,\infty)$, we have

$$\big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big)\ge 0$$

and

$$f(\tau)g(\tau)+f(\rho)g(\rho)\ge\ f(\tau)g(\rho)+f(\rho)g(\tau) \tag{2.2}$$

for all $\tau\le 0,\rho\le 0$. If we multiply both sides of the inequality (2.2) by $\tau+\frac{1}{\Gamma(\beta)}^{\beta-1}$ and integrate with respect to τ from 0 to t, we obtain

$$\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\big(f(\tau)g(\tau)\big)\,d\tau+\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ \big(f(\rho)g(\rho)\big)d\rho\ge$$
$$\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(\tau)g(\rho)d\tau+\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ f(\rho)g(\sigma)\,d\tau.$$

Using the following equality

$$\int_0^t f(\tau)d_{\beta}\tau=\int_0^t f(\tau)\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\tau$$

we have

$${}_0^{A}I_t^{\beta}\big(f(t)g(t)\big)+f(\rho)g(\rho){}_0^{A}I_t^{\beta}(1)\ge g(\rho){}_0^{A}I_t^{\beta}\big(f(t)+f(\rho)\big){}_0^{A}I_t^{\beta}\big(g(t)\big). \tag{2.3}$$

Multiplying both sides of the inequality (2.3) by $\rho+\frac{1}{\Gamma(\beta)}^{\beta-1}$ and integrating with respect to ρ on $[0,t]$, we have

$$\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}{}_0^{A}I_t^{\beta}(fg)(t)+\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)g(\rho){}_0^{A}I_t^{\beta}(1)\ \ge$$
$$\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}{}_0^{A}I_t^{\beta}(f)(t)g(\rho)+\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho){}_0^{A}I_t^{\beta}(g)(t)$$

then

$$ {}_{0}^{A}I_{t}^{\beta}(fg)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\rho+{}_{0}^{A}I_{t}^{\beta}(1)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)g(\rho)\,d\rho\geq{}_{0}^{A}I_{t}^{\beta}(f)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\rho+{}_{0}^{A}I_{t}^{\beta}(g)\,(t)\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)d\rho $$

$$ {}_{0}^{A}I_{t}^{\beta}(fg)(t){}_{0}^{A}I_{t}^{\beta}(1)+{}_{0}^{A}I_{t}^{\beta}(1)\,{}_{0}^{A}I_{t}^{\beta}(fg)(t)\geq{}_{0}^{A}I_{t}^{\beta}(f)(t){}_{0}^{A}I_{t}^{\beta}(g)(t)+{}_{0}^{A}I_{t}^{\beta}(g)(t){}_{0}^{A}I_{t}^{\beta}(f)(t) $$

$$ 2\,{}_{0}^{A}I_{t}^{\beta}(fg)(t){}_{0}^{A}I_{t}^{\beta}(1)\geq 2{}_{0}^{A}I_{t}^{\beta}(f)(t){}_{0}^{A}I_{t}^{\beta}(g)(t) $$

$$ {}_{0}^{A}I_{t}^{\beta}(fg)(t)\geq\frac{1}{{}_{0}^{A}I_{t}^{\beta}(1)}{}_{0}^{A}I_{t}^{\beta}(f)(t){}_{0}^{A}I_{t}^{\beta}(g)(t). $$

This completes the proof.

Theorem 2.2. *Let f and g be two synchronous functions on* $[0,1)$. *Then we have the following inequality*

$$ {}_{0}^{A}I_{t}^{\beta}(fg)(t){}_{0}^{A}I_{t}^{\beta}(1)+{}_{0}^{A}I_{t}^{\beta}(1){}_{0}^{A}I_{t}^{\beta}(fg)(t)\geq{}_{0}^{A}I_{t}^{\beta}(f)(t){}_{0}^{A}I_{t}^{\beta}(g)(t)+{}_{0}^{A}I_{t}^{\beta}(f)(t){}_{0}^{A}I_{t}^{\beta}(g)(t) \tag{2.4} $$

for all $t>0,\alpha>0,$ and $\beta>0$.

Proof Using the same way in the proof of Theorem 2.1, we can obtain (2.3). Multiplying both sides of the inequality (2.3) by $\rho+\frac{1}{\Gamma(\alpha)}^{\alpha-1}$ and integrating with respect to ρ from 0 to t we have

$$ {}_{0}^{A}I_{t}^{\beta}\,(fg)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}d\rho+ $$

$$ {}_{0}^{A}I_{t}^{\beta}\,(1)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}f(\rho)g(\rho)d\rho\geq $$

$$ {}_{0}^{A}I_{t}^{\beta}\,(f)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}g(\rho)d\rho+{}_{0}^{A}I_{t}^{\beta}\,(g)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}f(\rho)d\rho $$

$$ {}_{0}^{A}I_{t}^{\beta}\,(fg)(t){}_{0}^{A}I_{t}^{\alpha}\,(1)+{}_{0}^{A}I_{t}^{\beta}\,(1){}_{0}^{A}I_{t}^{\alpha}\,(fg)(t)\geq{}_{0}^{A}I_{t}^{\alpha}\,(f)(t){}_{0}^{A}I_{t}^{\beta}\,(g)(t)+{}_{0}^{A}I_{t}^{\beta}\,(f)(t){}_{0}^{A}I_{t}^{\alpha}\,(g)(t) $$

$$ 2{}_{0}^{A}I_{t}^{\beta}\,(fg)(t){}_{0}^{A}I_{t}^{\beta}\,(1)\geq 2{}_{0}^{A}I_{t}^{\beta}\,(f)(t){}_{0}^{A}I_{t}^{\beta}\,(g)(t) $$

$$ {}_{0}^{A}I_{t}^{\beta}\,(fg)(t)\geq\frac{1}{{}_{0}^{A}I_{t}^{\beta}\,(1)}{}_{0}^{A}I_{t}^{\beta}\,(f)(t){}_{0}^{A}I_{t}^{\beta}\,(g)(t) $$

and this ends the proof.

Remark 2.1. If the functions f and g are asynchronous on $[0,\infty)$, then the inequalities (2.2) and (2.3) are reversed.

Theorem 2.3. *We assume the functions f_i for $i=1,2,\dots,n$ are positive increasing functions on* $[0,\infty)$. *Then the following inequality holds*

$$ {}_{0}^{A}I_{t}^{\beta}\left(\prod_{i=1}^{n}f_i\right)(t)\geq\frac{1}{\left({}_{0}^{A}I_{t}^{\beta}(1)\right)^{n-1}}\prod_{i=1}^{n}\left({}_{0}^{A}I_{t}^{\beta}(f_i)\right)(t) \tag{2.5} $$

for any $t>0,\ \beta>0$.

Proof We use induction method to prove the theorem. We can easily see that for n = 1, we have

$$ {}_{0}^{A}I_{t}^{\beta}f_1(t)\geq{}_{0}^{A}I_{t}^{\beta}f_1(t) \tag{2.6} $$

for all $t>0,\ \beta>0$. For $n=2$, we use Theorem 2.1 and we have

$$ {}_{0}^{A}I_{t}^{\beta}\,(f_1f_2)(t)\geq\frac{1}{{}_{0}^{A}I_{t}^{\beta}(1)}{}_{0}^{A}I_{t}^{\beta}\,(f_1)(t){}_{0}^{A}I_{t}^{\beta}\,(f_2)(t). \tag{2.7} $$

We suppose that

$$ {}_0^A I_t^\beta \left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \frac{1}{\left({}_0^A I_t^\beta (1)\right)^{n-2}} \left(\prod_{i=1}^{n-1} {}_0^A I_t^\beta (f_i)\right)(t) \tag{2.8} $$

holds. Since the functions f_i, $i = 1,2,\dots,n$, are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is also an increasing function. If we choose $g(\mathrm{t}) \coloneqq \left(\prod_{i=1}^{n-1} f_i\right)(t)$, $f_n = f$ and use theorem 2.1, we obtain

$$ {}_0^A I_t^\beta \left(\prod_{i=1}^{n} f_i\right)(t) = {}_0^A I_t^\beta (gf)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (g)(t) {}_0^A I_t^\beta (f)(t). \tag{2.9} $$

Using the inequality (2.8) we have

$$ {}_0^A I_t^\beta \left(\prod_{i=1}^{n} f_i\right)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} \frac{1}{\left({}_0^A I_t^\beta (1)\right)^{n-2}} \left(\prod_{i=1}^{n-1} {}_0^A I_t^\beta (f_i)\right)(t) {}_0^A I_t^\beta (f_n)(t) \tag{2.10} $$

and this ends the proof.

Theorem 2.4. *Let f and g are two functions defined on $[0,\infty)$ such that f is increasing, g is differentiable. Assume that there exists a real number $m \coloneqq inf_{t\geq 0}\, \mathrm{g}'(t)$. Then for all $t > 0$ and $\beta > 0$, following inequality holds*

$$ {}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f(t)) {}_0^A I_t^\beta (g(t)) - \frac{m}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f(t)) {}_0^A I_t^\beta (t) + m {}_0^A I_t^\beta (tf(t)) \tag{2.11} $$

Proof We define the function *h(t) := g(t) - mt*. It is easy to see that the function h is increasing and differentiable on $[0,\infty)$. Using Theorem 2.1, we have

$$ {}_0^A I_t^\beta ((g(t) - mt) f(t)) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (g(t) - m(t)) {}_0^A I_t^\beta (f(t)) $$

$$ {}_0^A I_t^\beta (fg)(t) - m {}_0^A I_t^\beta (tf(t)) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f(t)) \left[{}_0^A I_t^\beta (g(t)) - m {}_0^A I_t^\beta (t)\right] $$

$$ {}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f(t)) {}_0^A I_t^\beta (g(t)) - \frac{m}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f(t)) {}_0^A I_t^\beta (t) + m {}_0^A I_t^\beta (tf(t)). $$

This concludes the prof.

Theorem 2.5. *Let the functions f and g are defined on $[0,\infty)$. We assume the function f is decreasing and g is differentiable. If there exists a real number M :=$sup_{t\geq 0}$ g' (t) Then we have*

$$ {}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta (g)(t) - \frac{M}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta t \tag{2.12} $$

for all $t > 0, \beta > 0$.

Proof We define $G(t) \coloneqq g(t) - Mt$. Since the function G is differentiable and decreasing on $[0,\infty)$, using Theorem 2.1 we have

$$ {}_0^A I_t^\beta (fG)(t) = {}_0^A I_t^\beta (f(t)(g(t) - Mt)) \geq \frac{1}{{}_0^A I_t^\beta (1)} \left[{}_0^A I_t^\beta (f(t)) {}_0^A I_t^\beta (g(t) - Mt)\right] \geq $$

$$ \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta (g)(t) - \frac{M}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta t. $$

Theorem 2.6. *Let the functions f and g are differentiable and there exists $m_1 : inf_{t\geq 0}\, g'(t)$. Then for all $t > 0$, $\beta > 0$ we have*

$$ {}_0^A I_t^\beta \left[\left(f(t)-m_1(t)\right)\left(g(t)-m_2(t)\right)\right] \geq \frac{1}{{}_0^A I_t^\beta(1)}\left[{}_0^A I_t^\beta\left(f(t)g(t)\right)-m_2\,{}_0^A I_t^\beta\left(tf(t)\right)-m_1\,{}_0^A I_t^\beta\left(tg(t)\right)+m_1m_2\,{}_0^A I_t^\beta\left(t^2\right)\right]. \quad (2.13) $$

Proof We consider the functions $F(t)=f(t)-m_1t$ *and* $G(t):=g(t)-m_2t$. It is clear that the functions $F(t)$ and $G(t)$ are increasing on $[0,\infty)$. Using Theorem 2.1, we have

$$ {}_0^A I_t^\beta[(f(t)-m_1t)(g(t)-m_2t)] = {}_0^A I_t^\beta\left(f(t)g(t)\right)-m_2\,{}_0^A I_t^\beta(f(t).t)-m_1\,{}_0^A I_t^\beta\left(t.g(t)\right)+m_1m_2\left({}_0^A I_t^\beta(t)\right)^2 $$

$$ \geq \frac{1}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta(f(t))\,{}_0^A I_t^\beta\left(g(t)\right)-\frac{m_2}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta\left(f(t)\right){}_0^A I_t^\beta(t)-\frac{m_1}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta(t)\,{}_0^A I_t^\beta\left(g(t)\right)+\frac{m_1m_2}{{}_0^A I_t^\beta(1)}\left({}_0^A I_t^\beta(t)\right)^2 $$

$$ \geq \frac{1}{{}_0^A I_t^\beta(1)}\left[{}_0^A I_t^\beta\left(f(t)\right){}_0^A I_t^\beta\left(g(t)\right)-m_2\,{}_0^A I_t^\beta\left(f(t)\right){}_0^A I_t^\beta(t)-m_1\,{}_0^A I_t^\beta(t)\,{}_0^A I_t^\beta\left(g(t)\right)+m_1m_2\left({}_0^A I_t^\beta(t)\right)^2\right]. $$

3. CONCLUSION

Fractional derivatives are an attraction point for several researchers. In this paper, we consider beta-fractional derivative. By using beta-fractional integrals, some new integral inequalities established in the case of two synchronous functions. As a main contribution to the literature, we prove six theorems. Our results are pioneer for the literature of integral inequalities in beta-fractional integral sense.

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