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A Taylor polynomial approach for solving generalized pantograph equations with nonhomogenous term

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A numerical method for solving the generalized (retarded or advanced) pantograph equation with constant and variable coefficients under mixed conditions is presented. The method is based on the truncated Taylor polynomials. The solution is obtained in terms of Taylor polynomials. The method is illustrated by studying an initial value problem. IIlustrative examples are included to demonstrate the validity and applicability of the technique. The results obtained are compared to the known results.

Keywords: advanced pantograph equation; Taylor series and polynomials

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1. Introduction

In recent years, pantograph equations have been studied by many authors who have investigated both their analytical and numerical aspects [14,3,16,6,5]. Functional differential equations with proportional delays are usually referred to as pantograpf equations or generalized equations. The name pantograph originated from the work of J.R.Ockendon and A.B.Tayler [18] on the collection of current by the pantograph head of an electric locomotive.

Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many different phenomena. In particular, they turn out to be fundamental when ODEs based models fail. These equations arise in many applications such as number theory [4], nonlinear dynamical system [15], industrial applications [10] and in studies based on biology, economy, control and electro-dynamic [1,2]

On the other hand, many methods based on Taylor polynomials have been given to find approximate solutions of the differential-difference and integro differential-difference equations [17,20,22,13,19,12,11]. Our purpose in this study is to develop and apply the mentioned methods to the pantograph equation with variable coefficients, which is an extension of pantograph equation given by Liu and Li [14] and Derfel and Iserles [3].

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In recent years there has been a growing interest in the numerical treatment of pantograph equations of the retarted and advanced type. A special feature of this type is the existance of compactly supported solitions [14]. This phenomonon was studied in [6], and has direct applications to approximation theory and wavelets [5].

The Taylor method has been shown to solve linear differential, integral, integro-differential equation equations and systems with approximate solutions which converge rapidly to accurate solutions [20,22,13,19]. The basic motivation of this study is to apply the Taylor method to the generalized pantograph equation

$$y^{(m)}(t) = \sum_{j=1}^{J} \sum_{k=0}^{m-1} P_{jk}(t) y^{(k)}(\alpha_j t) + f(t), \quad t \geq 0 \tag{1}$$

which is a generalization of the pantograph equations given by [14,3,6,5,9,16,1,2], under the initial conditions

$$\sum_{k=0}^{m-1} c_{ik} y^{(k)}(0) = \lambda_i; \quad i = 0, 1, \ldots, n-1 \tag{2}$$

and to find the solution in the truncated Taylor series form

$$y(t) = \sum_{n=0}^{N} y_n t^n, \quad y_n = \frac{y^{(n)}(0)}{n!}. \tag{3}$$

Here, $P_{jk}(t)$ and $f(t)$ are analytical functions; c_{ik}, λ_i and α_j are real or complex constants; the coefficitients y_n, $n = 0, 1, \ldots, N$ are Taylor coefficients to be determined.

2. Fundamental matrix relations

Let us convert the expressions defined in equations (1–3) to matrix forms. First, let us assume that the function $y(t)$ and its derivative $y^{(k)}(t)$, respectively, can be expanded to the Taylor series about $t = 0$ in the forms

$$y(t) = \sum_{n=0}^{\infty} y_n t^n, \quad y_n = \frac{y^{(n)}(0)}{n!} \tag{4}$$

and

$$y^{(k)}(t) = \sum_{n=0}^{\infty} y_n^{(k)} t^n \tag{5}$$

where for $k = 0$, $y^{(0)}(t) = y(t)$ and $y_n^{(0)} = y_n$.

Now, take the derivative of equation (5) with respect to t and then put $n \to n+1$:

$$y^{(k+1)}(t) = \sum_{n=1}^{\infty} n y_n^{(k)} t^{n-1} = \sum_{n=0}^{\infty} (n+1) y_{n+1}^{(k)} t^n \tag{6}$$

From (5), it is clear that

$$y^{(k+1)}(t) = \sum_{n=0}^{\infty} y_n^{(k+1)} t^n. \tag{7}$$

Using the relations (6) and (7), we have the recurrence relation between the Taylor coefficients $y_n^{(k)}$ and $y_n^{(k+1)}$ of $y^{(k)}(t)$ and $y^{(k+1)}(t)$:

$$y_n^{(k+1)} = (n+1)y_{n+1}^{(k)}; \quad n, k = 0, 1, 2, \ldots \tag{8}$$

Now, let us take $n = 0, 1, 2, \ldots, N$ and assume $y_n^{(k)} = 0$ for $n > N$. Then, the system (8) can be transformed into the matrix form

$$\mathbf{Y}^{(k+1)} = \mathbf{M}\mathbf{Y}^{(k)}, \quad k = 0, 1, 2, \ldots \tag{9}$$

where

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y_0^{(k)} \\ y_1^{(k)} \\ \vdots \\ Y_N^{(k)} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

For $k = 0, 1, 2, \ldots,$ it follows from (9) that

$$\begin{aligned} \mathbf{Y}^{(1)} &= \mathbf{M}\mathbf{Y}^{(0)} = \mathbf{M}\mathbf{Y} \\ \mathbf{Y}^{(2)} &= \mathbf{M}\mathbf{Y}^{(1)} = \mathbf{M}^2\mathbf{Y} \\ &\vdots \\ \mathbf{Y}^{(k)} &= \mathbf{M}\mathbf{Y}^{(k-1)} = \mathbf{M}^{\mathbf{k}}\mathbf{Y} \\ &\vdots \end{aligned} \tag{10}$$

where clearly

$$\mathbf{Y}^{(0)} = \mathbf{Y} = [y_0 \quad y_1 \quad \cdots \quad y_N]^T.$$

On the other hand, the solution expressed by equation (3) and its derivatives can be written in the matrix forms

$$[y(t)] = \mathbf{T}\mathbf{Y} \text{ and } [y^{(k)}(t)] = \mathbf{T}\mathbf{Y}^{(k)} \tag{11}$$

or using the relation in equation (10)

$$[y^{(k)}(t)] = \mathbf{T}\mathbf{M}^k\mathbf{Y} \tag{12}$$

where

$$\mathbf{T} = [1 \quad t \quad t^2 \quad \cdots \quad t^N].$$

To obtain the matrix form of the part

$$D(t) = \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\alpha_j t) \tag{13}$$

which is defined in equation (1), we first write the function $P_{jk}(t)$ in the form

$$P_{jk}(t) = \sum_{i=0}^{N} P_{jk}^{(i)} t^i, \; P_{jk}^{(i)} = \frac{P_{jk}^{:(i)}(0)}{i!} \tag{14}$$

and then, substituting (14) into (13), we obtain

$$D(t) = \sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N} P_{jk}^{(i)} t^i y^{(k)}(\alpha_j t). \tag{15}$$

Here it is seen that, from equation (5),

$$y^{(k)}(\alpha_j t) = \sum_{n=0}^{N} y_n^{(k)} (\alpha_j)^n t^n.$$

Hence the matrix representation of the terms $t^i y^{(k)}(\alpha_j t)$ in equation (15) becomes

$$[t^i y^{(k)}(\alpha_j t)] = \mathbf{T}\mathbf{I}_i\mathbf{A}_j\mathbf{Y}^{(k)}$$

or from (10),

$$[t^i y^{(k)}(\alpha_j t)] = \mathbf{T}\mathbf{I}_i\mathbf{A}_j\mathbf{M}^k\mathbf{Y}, \quad i = 0, 1, 2, \ldots, N \tag{16}$$

where

$$\mathbf{A}_j = \begin{bmatrix} (\alpha_j)^0 & 0 & \cdots & 0 \\ 0 & (\alpha_j)^1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (\alpha_j)^N \end{bmatrix}, \quad \mathbf{I}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

clearly $\mathbf{I}_i = \mathbf{I}_1^i$, $i = 0, 1, 2, \ldots, N$

Substituting the expression (16) into (15), we have the matrix relation

$$[D(t)] = \sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N} \mathbf{P}_{jk}^{(i)}\mathbf{T}\mathbf{I}_i\mathbf{A}_j\mathbf{M}^k\mathbf{Y}. \tag{17}$$

We now assume that the function $f(t)$ can be expanded as

$$f(t) = \sum_{n=0}^{N} f_n t^n, \quad f_n = \frac{f^{(n)}(0)}{n!}$$

or written in the matrix form

$$[f(t)] = \mathbf{T}\mathbf{F} \tag{18}$$

where

$$\mathbf{F} = [f_0 \quad f_1 \quad \cdots \quad f_N]^T.$$

Note that the matrix form of the first form $y^{(m)}(t)$, according to the relation (12), is

$$[y^{(m)}(t)] = \mathbf{T}\mathbf{M}^m\mathbf{Y} \tag{19}$$

Next we can obtain the corresponding matrix form for the initial conditions (7) as

$$\sum_{k=0}^{m-1} c_{ik}\mathbf{T}(0)\mathbf{M}^k\mathbf{Y} = [\lambda_i]; \quad i = 0, 1, 2, \ldots, m-1 \tag{20}$$

where

$$\mathbf{T}(0) = [1 \quad 0 \quad 0 \quad \cdots \quad 0].$$

3. Method of solution

We are now ready to construct the fundamental matrix equation corresponding to equation (1). For this purpose, substituting the matrix relations (17–19) into equation (1) and then simplifying, we obtain the fundamental matrix equation

$$\left\{\mathbf{M}^m - \sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N}\mathbf{P}_{jk}^{(i)}\mathbf{I}_i\mathbf{A}_j\mathbf{M}^k\right\}\mathbf{Y} = \mathbf{F} \tag{21}$$

which corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $y_0, y_1, \ldots, y_n$.

Briefly, we can write equation (21) in the form

$$\mathbf{WY} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}] \tag{22}$$

so that

$$\mathbf{W} = [w_{pq}] = \mathbf{M}^m - \sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N}\mathbf{P}_{jk}^{(i)}\mathbf{I}_i\mathbf{A}_j\mathbf{M}^k.$$

Also, the matrix form (20) for the conditions (2) can be written as

$$\mathbf{U}_i\mathbf{Y} = [\lambda_i] \text{ or } [\mathbf{U}_i; \lambda_i]; \quad i = 0, 1, 2, \ldots, m-1 \tag{23}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1} c_{jk}\mathbf{T}(0)\mathbf{M}^k = [u_{i0} \quad u_{i1} \quad \cdots \quad u_{iN}].$$

To obtain the solution of equation (1) under the initial conditions (2), by replacing the – rows matrices $[\mathbf{U}_i; \lambda_i]$ in equation (23) by the last m rows of the matrix $[\mathbf{W}; \mathbf{F}]$ in equation (22), we have the argumented matrix [20,22]

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{F}}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f_{N-m} \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix} \tag{24}$$

If $\det(\tilde{\mathbf{W}}) \neq 0$, then we can write

$$\mathbf{Y} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{F}}. \tag{25}$$

Thus, the coeffientients y_n, $n = 0, 1, \ldots, N$ are uniquely determined by (25). If $\det(\tilde{\mathbf{W}}) = 0$, then there is no solution and the method can not be used. Also, by means of systems we may obtain the particular solution.

On the other hand, we can easily check the accuracy of the solutions as follows [22,13]:

Since the obtained polynomial solution is an approximate solution of equation (1), it must be satisfied approximately; that is, for $t = t_r$, $r = 0, 1, 2, \ldots$

$$E(t_r) = \left| y^{(m)}(t_r) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t_r) y^{(k)}(\alpha_j t_r) - f(t_r) \right| \cong 0$$

or

$$E(t_r) \leq 10^{-k} \ (k_r \text{ positive integer}).$$

If $\max 10^{-k_r} = 10^{-k}$ (k any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_r)$ at each of the points t_r becomes smaller than the prescribed 10^{-k}.

4. Examples

In this section, several numerical example are given to illustrate the properties of the method and all of them were performed on the computer using a program written in Maple9. The absolute errors in Tables are the values of $|y(t) - y_N(t)|$ at selected points.

Example 1 [9] Let us first consider the equation

$$y'(t) = \frac{1}{2} \mathrm{e}^{(t/2)} y\left(\frac{t}{2}\right) + \frac{1}{2} y(t), \quad 0 \leq t \leq 1, \quad y(0) = 1 \tag{26}$$

which has the exact solution $y(t) = \mathrm{e}^{\mathrm{t}}$. When the presented method is applied to equation (24), the fundamental matrix equation becomes

$$\left\{ \mathbf{M}^1 - \sum_{j=0}^{1}\sum_{k=0}^{0}\sum_{i=0}^{5} \mathbf{P}_{jk}^{(i)} \mathbf{I}_i \mathbf{A}_j \mathbf{M}^k \right\} \mathbf{Y} = \mathbf{F}.$$

Hence, the computed results are compared with other methods [1,7,21,8] in Table 1.

Figure 1 shows the plot of the error points $E(t_i)$ for $N = 5$, $N = 7$ and $N = 9$. This plot clearly indicates that when we increase the truncation limit N, we have less error.

Example 2 Consider the pantograph equation of third order

$$y'''(t) = t y''(2t) - y'(t) - y\left(\frac{t}{2}\right) + t\cos(2t) + \cos\left(\frac{t}{2}\right)$$

$$y(0) = 1, y'(0) = 1, y''(0) = -1.$$

We give numerical analysis for various N values in Table 2.

Table 1. Error analysis of Example 1 for the t value.

t	Spline Fnk. Approx. [7]	Present method ($N=5$)	Spline method [21]	Spline method [8]	Present method ($N=7$)	ADM with 13 terms [1]	Present method ($N=9$)
0.2	0.198 E-7	0.271E-6	1.37 E-11	3.10 E-15	0.256E-10	0.000000	0.705E-14
0.4	0.473 E-7	0.882E-5	3.27 E-11	7.54 E-15	0.333E-8	2.22 E-16	0.106E-10
0.6	0.847 E-7	0.682E-4	5.86 E-11	1.39 E-14	0.577E-7	2.22 E-16	0.294E-9
0.8	0.135 E-6	0.293E-3	9.54 E-11	2.13 E-14	0.438E-6	1.33 E-15	0.386E-8
1	0.201 E-6	0.912E-3	1.43 E-10	3.19 E-14	0.212E-5	4.88 E-15	0.290E-7

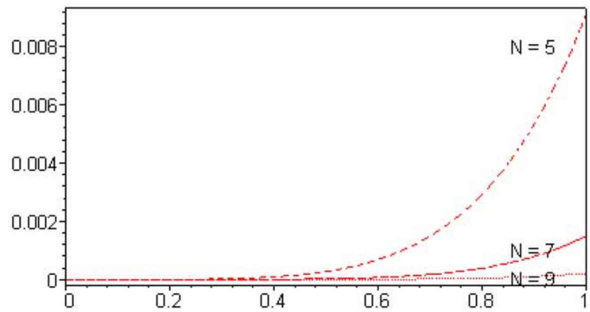

Figure 1. Error points E(t_i) and the truncation limits for $N = 5$, 7 and 9.

Table 2. Error analysis of Example 2 for the t value.

t	Exact solution	Present method $N_e = 5$	Present method $N_e = 8$	Present method $N_e = 10$
0.0	1.000000	0.000000	0.000000	0.000000
0.2	0.980066	0.112E−4	0.100E−11	0.900E−11
0.4	0.921060	0.398E−4	0.132E−7	0.115E−9
0.6	0.825335	0.134E−3	0.574E−6	0.986E−8
0.8	0.696706	0.115E−2	0.786E−5	0.234E−6
1.0	0.540302	0.417E−2	0.587E−4	0.271E−5

Example 3 [16] Consider the pantograph equation of first order

$$y'(t) = -y(t) + \frac{q}{2}y(qt) - \frac{q}{2}e^{-qt}, \quad y(0) = 1 \tag{28}$$

where $y(t) = e^{-t}$. Table 3 compares the results of the present method and collocation method [20] for this problem.

Example 4 [9] Consider the pantograph equation of second order

$$y''(t) = \frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2, \quad y(0) = 0, \quad y'(0) = 0, \ 0 \le t \le 1.$$

After the ordinary operations and following the method in Section 3, we obtain $y(t) = t^2$ and this is the exact solution.

Table 3. Error analysis of Example 3 for the t value.

t	Muroya [16] ($q = 1$)	Present method $N = 10$	$N = 11$	$N = 12$	Muroya [16] ($q = 0.2$)	Present method $N = 10$	$N = 12$
2^{-1}	0.500E−5	0.100E−9	0.300E−9	0.000000	0.219E−4	0.200E−9	0.124E−9
2^{-2}	0.187E−6	0.200E−9	0.500E−9	0.100E−9	0.108E−5	0.100E−9	0.974E−10
2^{-3}	0.643E−8	0.200E−9	0.400E−9	0.000000	0.381E−7	0.100E−9	0.700E−10
2^{-4}	0.210E−9	0.200E−9	0.400E−9	0.000000	0.126E−8	0.100E−9	0.914E−10
2^{-5}	0.670E−11	0.100E−9	0.200E−9	0.000000	0.409E−10	0.100E−9	0.528E−10
2^{-6}	0.210E−12	0.000000	0.100E−9	0.000000	0.120E−11	0.000000	0.195E−10

Example 5 Let us consider the problem

$$y''(t) = \frac{1}{2}ty'(2t) - 3y\left(\frac{1}{2}t\right) - \frac{15}{4}t^2 + 2t - 3, \quad y(0) = -1, \quad y'(0) = 2. \tag{30}$$

Following the previous procedures, we get the approximate solution of problem (30) for $N = 5$ as

$$y(t) = 3t^2 + 2t - 1$$

which is an exact solution.

5. Conclusions

A new technique using the Taylor series to numerically solve the pantograph equations is presented. Nonhomogenous pantograph equation with variable coefficients are usually difficult to solve analyticaly. Then it is required to obtain the approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

It is observed that the method has the best advantage when the known functions in an equation can be expanded to the Taylor series with converge rapidly. To get the best approximation, we take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen large enough.

The method presented in this study is a method for computing the coefficients in the Taylor expansion of the solution of a nonhomogenous pantograph equation.The Taylor matrix method is an effective method for cases where the known functions have the Taylor series expansions at $t = 0$. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial of degree N or less than N.

The method can also be extended to the non-linear pantograph initial problem but some modifications are required.

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