S.Ü. Fen-Edebiyat Fakültesi Fen Dergisi Sayı:19 (2002) 71-79 KONYA

On a LOD and Crank-Nicolson Methods For the Two Dimensional Diffusion Equation

Mustafa GÜLSU[1]

Abstract: *Finite-difference techniques based on explicit method and Crank-Nicolson method for one dimensional diffusion are used to solve the two-dimensional time dependent diffusion equation with boundary conditions. In these cases locally one-dimensional (LOD) techniques are used to extend the one-dimensional techniques to solve the two-dimensional problem. The results of numerical testing show that these schemes use less central processor (CPU) time than the fully implicit scheme.*

Key Words: **Finite-difference , diffusion equation, Crank-Nicolson Method, LOD.**

İki Boyutlu Diffuzyon Denklemi için Yerel Bir Boyut ve Crank-Nicolson Metodları Üzerine

Özet: *Bu çalışmada bir boyutlu diffuzyon denklemi için Açık yöntem ve Crank Nicolson yöntemini temel alan sonlu fark teknikleri , iki boyutlu zamana bağımlı diffuzyon denklemini çözmek için kullanıldı. Yerel bir boyut(LOD) yöntemi iki boyutlu diffuzyon denklemini çözmek için genişletildi. Nümerik sonuçlar ile bu yöntemin kapalı yöntemlere göre daha az zaman (CPU) harcadığı gösterildi.*

Anahtar Kelimeler: **Sonlu farklar, difizyon denklemi, Crank-Nicolson Yöntemi, Yerel Bir Boyut Yöntemi**

1-Introduction

The constant-coefficient two-dimensional diffusion equation, namely

$$\frac{\partial u}{\partial t}=\alpha_x\frac{\partial^2 u}{\partial x^2}+\alpha_y\frac{\partial^2 u}{\partial y^2}\quad,\quad 0\le x\le M\,,\ 0\le y\le N\,, \tag{1.1}$$

$$0\le t\le T$$

[1] Muğla University, Department of Mathematics, [48000] Muğla-Turkey

where α_x and α_y are the coefficients of diffusion in the x and y directions respectively, has many applications to practical problems, including the flow of groundwater, and the diffusion of heat through solids. For many years the standart explicit two-level finite difference method for solving (1.1) was the classical explicit forward-time centred-space method described in Noye B.J., Hayman K.J.[4]

Recent improvements include the efficient alternating group explicit method of Dehghan M. [2]. The present article investigate the development of a fourth-order accurate two-level explicit finite difference method for solving (1.1) subject to Drichlet boundary condition. In particular locally one dimensional (LOD) method and Crank-Nicolson method are investigated.

For convenience, a method which uses a computational molecule that involves m_1 grid points from time level (n+1) and m_2 grid points from time level n is denoted as an (m_1,m_2) methods. Also, the grid point $(i\Delta x, j\Delta y, n\Delta t)$ i=0,1,2,...I, j=0,1,2,...,J, n=0,1,2,...K where $\Delta x=M/I$, $\Delta y=N/J$, $\Delta t=T/K$, is referred to as the (i,j,n) grid point. At this point the partial differential equation (PDE) (1.1) is discretised to give the aproximating finite difference equation (FDE)

$$\sum_l \sum_m a_{l,m} u_{i+l,j+m}^{\,n+1} = \sum_l \sum_m b_{l,m} u_{i+l,j+m}^{\,n} \qquad 1.2$$

The coefficient $a_{l,m}$ and $b_{l,m}$ are functions of the non dimensional diffusion numbers

$$r_x = \alpha_x \frac{\Delta t}{(\Delta x)^2}, \qquad r_y = \alpha_y \frac{\Delta t}{(\Delta y)^2}$$

Theoretical comparisons of the order of convergence of various finite-difference methods are based on the leading error terms in their modified equivalent partial differential equations (MEPDE) which have the general form

$$\frac{\partial u}{\partial t} - \alpha_x \frac{\partial^2 u}{\partial x^2} - \alpha_y \frac{\partial^2 u}{\partial t^2} + \sum_{p=3}^{\infty} \sum_{q=0}^{p} C_{p,q} \frac{\partial^p}{\partial x^{p-q} \partial y^q} = 0 \qquad 1.3$$

where the $C_{p,q}$ are coefficients of errors term. Given that (1.2) is consistent with the two-dimensional diffusion equation (1.1) which requires that

$$Lim_{\Delta x,\Delta y,\Delta t \to 0} C_{p,q} = 0 \qquad \text{for } p \geq 0, \qquad 1.4$$

the error coefficient $C_{p,q}$ in the MEPDE can be written in the form;

$$C_{p,q}=\begin{cases}\dfrac{2\alpha_x(\Delta x)^{p-2}}{p!}\Gamma_{p,q}(r_x,r_y).....q=0\\ \dfrac{2\alpha_y(\Delta y)^{p-2}}{p!}\Gamma_{p,q}(r_x,r_y).....q=p\\ \dfrac{4\alpha_x(\Delta x)^{p-q-2}(\Delta y)^q}{(p-q)!q!}\Gamma_{p,q}(r_x,r_y).....otherwise\end{cases} \qquad 1.5$$

It can be seen from (1.3) that the error term associated with the coefficients $C_{p,q}$ are of the order (p-2) in Δx and Δy. The order of accuracy of an FDE which approximately solves (1.1) is the smallest order of any error term present in the corresponding MEPDE. Hence if the leading error term in the MEPDE is $C_{P,q}$ for any q=0,1,2,...,P then the FDE is order (P-2) accurate.[4]

In the following the time-stepping stability of the FDE (1.2) is established by means of the von Neumann method.

In order to verify theoretical predictions, numerical tests were carried out on a two dimensional time-dependent diffusion equation:

$$\frac{\partial u}{\partial t}=\alpha_x\frac{\partial^2 u}{\partial x^2}+\alpha_y\frac{\partial^2 u}{\partial y^2} \qquad 1.6$$

$u(x,y,0)=f(x) = \exp(x+y) \qquad 0\le x\le 1, 0\le y\le 1$ 1.7

$u(0,y,t)=g_0(y,t)= \exp(y+2t)\ , \qquad 0\le t\le T, 0\le y\le 1$

$u(1,y,t)= g_1(y,t)= \exp(1+y+2t), \quad 0\le t\le T, 0\le y\le 1$

$u(x,1,t)= h_1(x,t)= \exp(1+x+2t), \quad 0\le t\le T, 0\le x\le 1$

$u(x,0,t)= h_0(x,t)= \exp(x+2t), \qquad 0\le t\le T, 0\le x\le 1$

2. LOD Methods

Partial Differential Equation (1.1) can be solved by splitting it into two one-dimensional equation

$$\frac{1}{2}\frac{\partial u}{\partial t}=\alpha_x\frac{\partial^2 u}{\partial x^2} \qquad 2.1a$$

$$\frac{1}{2}\frac{\partial u}{\partial t}=\alpha_y\frac{\partial^2 u}{\partial y^2} \qquad 2.1b$$

rather than discretising the complete two-dimensional diffusion equation to give an approximating finite-difference equation based on a two-dimensional computational molecule. Each of these equations is then solved over half of the time step used for the complete two-dimensional equation using techniques for the one dimensional problems. This is advantageous since accurate and stable techniques for one -dimensional diffusion are much easier to develop and use than single step methods for two-dimensional diffusion equation.

Commencing with the initial condition for each n=0,1,2,...,K the process of stepping from time t_n to t_{n+1} is carried out in two stages. In the first stage , in advancing from t_n=nk to the time $t_{n+\frac{1}{2}} = (t_n + \frac{k}{2})$, the partial differential equation

$$\frac{1}{2}\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} \qquad 2.2$$

is solved numerically at the spatial points (x_i, y_j) , i=1,2,...,I-1 for each j=0,1,...,J.

Commencing with previously computed values $u_{i,j}^{\ n}$ i,j=1,2,...,M-1 and boundary values:

$$u_{0,j}^{\ n} = g_0(y_j, t_n) \quad , \quad j=0,1,2,...,J \qquad 2.3$$

$$u_{M,j}^{\ n} = g_1(y_j, t_n) \quad , \quad j=0,1,2,...,J$$

results in the set of approximate values $u_{i,j}^{\ n+\frac{1}{2}}$, i=1,2,...,I-1 , j=0,1,...,J being found at the intermediate time $t_{n+\frac{1}{2}}$.

Then in advancing from the time $t_{n+\frac{1}{2}}$ to $t_{n+1} = (t_n + k)$ the equation:

$$\frac{1}{2}\frac{\partial u}{\partial t} = \alpha_y \frac{\partial^2 u}{\partial y^2} \qquad 2.4$$

is solved numericaly at the spatial points (x_i, y_j) , commencing with initial values $u_{i,j}^{\ n+\frac{1}{2}}$, i=1,2,...,I-1 , j=1,2,...,J-1 and using as boundary values $u_{i,0}^{\ n+\frac{1}{2}}$ and $u_{i,M}^{\ n+\frac{1}{2}}$ i=1,2,...,I-1. Not that the boundary conditions (1.7) are not used at the intermediate time $t_{n+\frac{1}{2}}$. This is because in the time interval t_n to $t_{n+\frac{1}{2}}$, the process of diffusion in the x-direction has been applied with a diffusion coefficient which is twice that in the original equation (1.1) as can be seen by rearranging in the form

$$\frac{\partial u}{\partial t} = 2\alpha_x \frac{\partial^2 u}{\partial x^2} \qquad 2.5$$

Not that the values of $u_{i,j}^{n+\frac{1}{2}}$ i=1,2,...,I , j=1,2,...,J are not approximate solutions to the original problem.

Let's running the LOD process using explicit method for which the correct two-stage procedure is:

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{k} = \frac{1}{h^2}\left\{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}\right\} \qquad 2.6$$

for each j=0,1,2,...,J apply

$$u_{i,j}^{n+\frac{1}{2}} = r_x\left(u_{i-1,j}^{n} + u_{i+1,j}^{n}\right) + (1-2r_x)u_{i,j}^{n} \qquad 2.7$$

for each i=1,2,...,I-1 then for each i=1,2,...,I-1 apply

$$u_{i,j}^{n+1} = r_y\left(u_{i,j-1}^{n+\frac{1}{2}} + u_{i,j+1}^{n+\frac{1}{2}}\right) + (1-2r_y)u_{i,j}^{n+\frac{1}{2}} \qquad 2.8$$

for each j=1,2,...,J-1. These are von Neumann stable for $0 < r_x \le \frac{1}{2}$, $0 < r_y \le \frac{1}{2}$.[5]

If $r_x=r_y=r^*=1/6$ the results obtained should be fourth-order accurate and if $r_x=r_y=r^*=1/2$ the results should be second –order accurate. However , when known boundary values $u_{i,0}^{n+\frac{1}{2}}$, $u_{i,j}^{n+\frac{1}{2}}$, i=0,1,...,I for the complete problem computed using (1.7) are used instead of those calculated using (2.7) the result shown in Figure1. indicate that this LOD procedure has produced only second order results, for s*=1/6 the slope of the line of best fit which gives an estimate of the order of convergence of the error, is 1.82, and for r*=1/2 it is 2.02.

The correct boundary values to be used along y=0 and y=N at the intermediate time level for the second half-time step are those obtained from the boundary values at the previous time t_n by applying the one-dimensional finite-difference equation being used elsewhere in the interior of the region . Note that end-points values along x=0 and x=M at the intermediate time level are not required in the second stage.

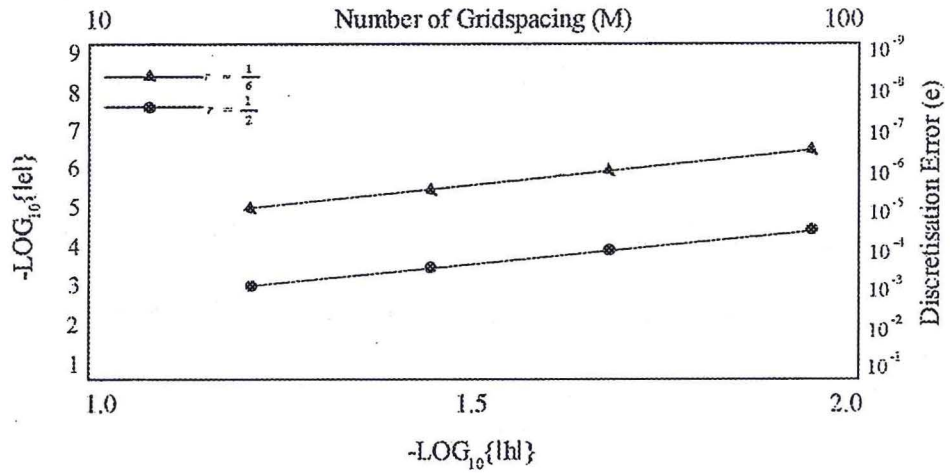

Figure1 Relation between error u and gridspacing for exp.method with using (1.7)

The numerical result obtained with this procedure are shown in Figure2. It is clear that the errors when r*=1/6 are now of order fourth. In fact , the slope of the line of best fit for r*=1/6 is 4.01 while that for r*=1/2 is 2.02. This clearly shows that the correct treatment of the boundaries at the intermediate time level for any time-splitting procedure is very important in the generation of the final solution.

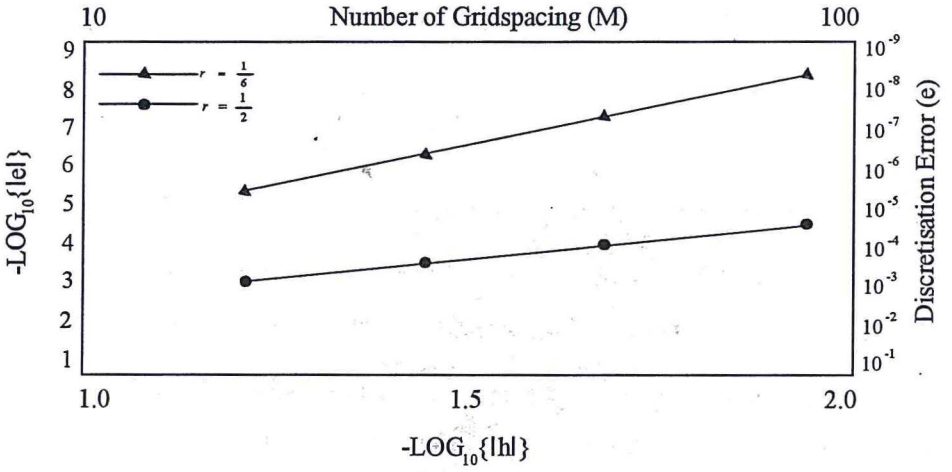

Figure2 Relation between error u and gridspacing for exp.method with using (2.7)

3.Crank-Nicolson Method

The Crank-Nicolson method is used in the following. Equation (2.1a) is solved numerically over the half-time step t_n to $t_{n+1/2}$ using:

$$-r_x u_{i,j-1}^{n+\frac{1}{2}} + 2(1+r_x)u_{i,j}^{n+\frac{1}{2}} - r_x u_{i,j+1}^{n+\frac{1}{2}} = r_x u_{i,j-1}^{n} + 2(1-r_x)u_{i,j}^{n} - r_x u_{i,j+1}^{n} \qquad 3.1$$

applied for j=2,3,...,J-2 for each i=0,1,...,I. This procedure is unconditionally von Neumann stable and solvable for all r>0 . Then for each i=0,1,...,I values at points adjacent to the boundary y=0 are calculated using the forward time centered space (FTCS) formula in the following form:

$$u_{i,1}^{n+\frac{1}{2}} = r_x u_{i,0}^{n} + (1-2r_x)u_{i,1}^{n} + r_x u_{i,2}^{n} \qquad 3.2$$

while values at points adjacent to the boundary y=1 are calculated using

$$u_{i,M-1}^{n+\frac{1}{2}} = r_x u_{i,M}^{n} + (1-2r_x)u_{i,M-1}^{n} + r_x u_{i,M-2}^{n} \qquad 3.3$$

Both (3.2) and (3.3) are stable only for $0<r\leq 1/2$.[2]

The Cranc-Nicolson formula is then used to solve Eq.(2.1b) over the time interval $t_{n+\frac{1}{2}}$ to t_{n+1} as follows. For i=1,2,...,I-1 and each j=j=1,2,...,J-1 use :

$$-r_y u_{i-1,j}^{n+1} + 2(1+r_y)u_{i,j}^{n+1} - r_y u_{i+1,j}^{n+1} = r_y u_{i-1,j}^{n+\frac{1}{2}} + 2(1-r_y)u_{i,j}^{n+\frac{1}{2}} + r_y u_{i+1,j}^{n+\frac{1}{2}} \qquad 3.4$$

The formula (3.1) (3.2) and (3.3) are used for i=0 and i=M , so the values of $u_{0,j}^{n+\frac{1}{2}}$ and $u_{M,j}^{n+\frac{1}{2}}$, j=0,1,...,J which are required in using formulae (3.4), have already been found. Values of $u_{i,j}^{n+1}$ on the boundaries x=0,1 and y=0,1 for the local problem are provided by the boundary conditions (1.7).
This procedure is unconditionaly von Neumann stable , and solvable for all r>0.
When the absolute value of the error;

$$e_{i,j}^{n} = u(ih, jk, nk) - u_{i,j}^{n} \qquad 3.5$$

at the point (0.5,0.5) at time T=1.0 was graphed against h on a logarithmic scale for various r , it was found that the slopes of lines were always close to 2 for Cranc-Nicolson formula and the explicit formula. These results illustrate the theoretical orders of accuracy evident from the modified equivalent equation.

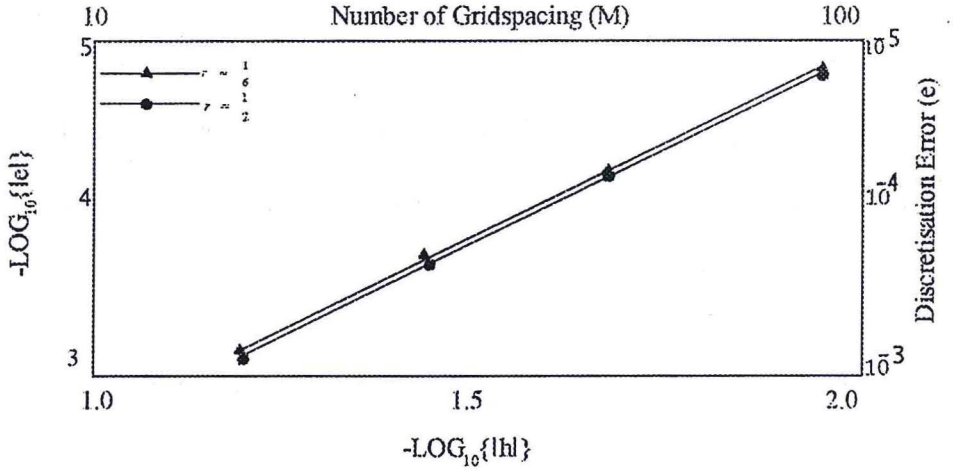

Figure3 Relation between error u and gridspacing for Crank-Nicolson method

x	y	Exp. Method	Cr-Nic Method	Exp.- Error	Cr-Nic -Error	Analitical Solutions
0.1	0.1	9.024880145	9.024953	0.00133354	$-0.6x10^{-4}$	9.025013499
0.2	0.2	11.02277750	11.022976	0.00039888	$-0.2x10^{-3}$	11.02317638
0.3	0.3	13.46303642	13.463438	0.00070162	$-0.3x10^{-3}$	13.46373804
0.4	0.4	16.44367357	16.444147	0.00097320	$-0.5x10^{-3}$	16.44464777
0.5	0.5	20.08438023	20.904643	0.00115669	$-0.6x10^{-3}$	20.08553692
0.6	0.6	24.53132433	24.531930	0.00120587	$-0.6x10^{-3}$	24.53253020
0.7	0.7	29.96301156	29.963600	0.00108849	$-0.5x10^{-3}$	29.96410005
0.8	0.8	36.59744019	36.597834	0.00079425	$-0.4x10^{-3}$	36.59823444
0.9	0.9	44.70082306	44.700858	0.00036143	$-0.1x10^{-3}$	44.70118449

Table1 Results for u with T=1.0 , h=0.05, r=1/2

4.Conclusion

In this paper two-methods namely the explicit method and the second-order Crank-Nicolson method are used to solve the two-dimensional diffusion equation with boundary condition through a LOD procedure which employed those one-dimensional schemes to apply them in each direction. Using the explicit method for one-dimensional diffusion equation in a LOD procedure with special treatment on the boundaries at the intermediate time level gave fourth-order accuracy. Without the special boundary treatment at the intermediate time levels high-order methods used at interior grid points in an LOD procedure only produce low-order results.

A comparison with the fully implicit schemes for the model problem clearly demonstrates that the new techniques use less CPU time. The only disadvantage of these methods was their limited range of stability. This was because of avoiding the use of the

boundary values at the intermediate time levels as this makes these procedures to be dependent to some other conditional schemes to evaluate the values near the boundaries.

Also Crank-Nicolson method produced second-order results. It used more CPU time than the fourth-order LOD procedure to get results of the same accuracy.

References

1- Ames, W.F., **"Numerical Methods for Partial Differential Equations",** Academic Press, New York,(1992)

2- Dehghan,M., **"Implicit Locally One-Dimensional Methods for Two-Dimensional Diffusion with a Non-Local Boundary Condition"**, Math. And Computers in Simulation,Vol.49, No.4,331p-349p.,(1999)

3- Herceg,D.,Cvetkovic,L., **"On a Numerical Differentiation"**, SIAM J. Numer.Anal.,Vol.23, No.3, 686p-691p.,(1986)

4- Noye,B.J.,Hayman,K.J., **"Explicit Two-Level Finite –Difference Methods for The Two-Dimensional Diffusion Equation"**, Intern.J.Computer Math.,Vol.42, 223p-236p.,(1991)

5- Smith, G. D., **"Numerical Solution of Partial Differantial Equations"**, Oxford Univ. Press, Oxford.,(1985)

6- Vanaja, V., Kellogg, R.B., **"Iterative Methods for a Forward-Backward Heat Equation"**, SIAM J.Numer.Anal., Vol.27, No.3, 622p-635p.,(1990)