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ORIGINAL RESEARCH

An operational matrix method to solve linear Fredholm–Volterra integro-differential equations with piecewise intervals

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Abstract

In recent times, operational matrix methods become overmuch popular. Actually, we have many more operational matrix methods. In this study, a new remodeled method is offered to solve linear Fredholm–Volterra integro-differential equations (FVIDEs) with piecewise intervals using Chebyshev operational matrix method. Using the properties of the Chebyshev polynomials, the Chebyshev operational matrix method is used to reduce FVIDEs into a linear algebraic equations. Some numerical examples are solved to show the accuracy and validity of the proposed method. Moreover, the numerical results are compared with some numerical algorithm.

Keywords Piecewise Fredholm–Volterra equations · Operational matrix method · Chebyshev polynomials

Introduction

Integro-differential equations are very important to model a real world phenomenons. Integro-differential equations usually are a combination of differential, Fredholm and Volterra integral equations. These type of equations arise in applied sciences such as wave mechanics, heat conduction, medicine, chemistry, astronomy, electrostatics, etc.[1–4]. Hence, the solutions of these type equations gain prominence to find out the behavior of modeling.

These type of equations usually difficult to solve exactly since it has many parts of differential, Fredholm and Volterra integral. The Fredholm–Volterra integro differential equations (FVIDEs) have been widely studied by many more authors to obtain the numerical solutions. In [5], authors introduced an efficient Bernoulli matrix method to solve high order linear Fredholm integro differential equation with piecewise intervals. In [6], an efficient Bernoulli collocation method has been developed to gain numerical solution such an equations. In [7], Acar and Daşçıoğlu developed a projection method based on Bernstein polynomials for solution of linear FVIDEs. In [8], Kürkçü, Aslan and Sezer presented a collocation method using hybrid Dickson and Taylor polynomials to obtain the numerical solutions of FVIDEs.

In [9], Yüksel et al. obtained a Chebyshev polynomial method for high-order linear Fredholm–Volterra integro-differential equations. In [10], Ebrahimi and Rashidinia produced a cubic B-spline approach by using the Newton–Cotes formula for FVIDEs. Also, we have many more studies in literature such as Dickson polynomials solution [11], Lucas polynomials solution [12], a polynomial solution [13], the backward substitution method [14], He's homotopy perturbation method [15], Mott polynomials solution [16], Laguerre polynomial solution [17], Taylor series solution [18], the semi orthogonal B-spline wavelet solution [19], a Tau method [20] and the power series method [21].

In this study, a operational matrix method is presented to solve the linear FVIDEs with piecewise intervals of the following form

$$\sum_{k=0}^{m} P_k(x)y^{(m)}(x) + \sum_{m=0}^{p} \mu_m \int_{c_m}^{x} V_m(x,t)y(t)\mathrm{d}t + \sum_{n=0}^{q} \lambda_n \int_{a_n}^{b_n} F_n(x,t)y(t)\mathrm{d}t = f(x) \tag{1}$$

with mixed conditions

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$$\sum_{k=0}^{m-1}\left(a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)+c_{ik}y^{(k)}(c)\right)=\alpha_i,\quad i=0,1,\dots,m-1. \tag{2}$$

where the parameter λ_s, μ_r and λ_s are constants.$P_k(x)$ and $f(x)$ are known and belong to $L^2[0,1]$. We desire to find the unknown function $y(x)$. For this purpose, the approximation series are defined by

$$y_N(x)=\sum_{r=0}^{N}a_rT_r^*(x),\quad x\in[0,1], \tag{3}$$

where N is any positive integer and $T_r^*(x)$, $r=0,1,\dots,N$ denote the shifted Chebyshev polynomials [22] and $N\geq m$.

This paper is organized as follows: Definitions and some properties of the Chebyshev polynomials are mentioned in Sect. 2. Section 3 is introduced representation of the matrix form of differential, Fredholm and Volterra integral part in Eq. (1). The numerical method establishes in Sect. 4. In Sect. 5, several treatments are presented. In Sect. 6, a conclude adds the paper. All computations have been calculated by Maple13. Figures have been plotted by Matlab.

The operational matrix method has been investigated by some author [23–29]. In these studies, this method is successfully solved the Abel equation, fractional integro differential equations, the Lane-Emden equation, fractional order differential equations and nonlinear Volterra integro differential equations. All above issues motivate us to introduce an operational matrix method for FVIDEs.

Chebyshev polynomials

It is well known that the fundamental theorem of approximation is called Weierstrass Theorem which says us any continuous function can be approximated uniformly by polynomials (See [30] for details). If you need a polynomial to make an approximation, you should choose an ordinary Fourier series (See [31] for details). The first type Chebyshev polynomial is a Fourier cos series.

Describe $T_n(x)$ which is called the Chebyshev functions family by formula, for $n\geq 0$

$$T_n(x)=\cos(n\,\mathrm{arc}\,cosx),\quad x\in[-1,1]$$

Theorem 1 *The family of $T_n(x)$ satisfy the following properties.*

a. *The degree of $T_n(x)$ is n.*
b. *For $n\geq 1$, $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$.*
c. $T_n(x)=2^{n-1}x^n+\cdots$
d. $(T_i,T_j)_w=\int_{-1}^{1}w(x)T_i(x)T_j(x)\mathrm{d}x=0$, $i\neq j$ *where* $w(x)=(1-x^2)^{-1/2}$ *is called weight function.*
e. *The roots of the $T_{n+1}(x)$ are $x_k=\cos\left(\frac{(2k+1)\pi}{2(n+1)}\right)$, $0\leq k\leq n$ which is called the Chebyshev nodes to compute interpolating approximations for continuous functions.*

Proof: See [22].

Theorem 2. *Let $f\in C^{n+1}[-1,1]$ and the n degree polynomial $p_n(x)$ interpolate to f. Using the Chebyshev nodes, we have.*

$$\left\|f-p_n\right\|_\infty\leq\frac{1}{2^n(n+1)!}\left\|f^{(n+1)}\right\|_\infty \tag{4}$$

Proof: See [22].

Theorem 3. *Let $y_N(x)$ be an approximation to $y(x)$. The truncation error $E_T(N)$ can be bounded by the following inequality. If.*

$$y_N(x)=\sum_{r=0}^{N}a_rT_r^*(x) \tag{5}$$

then

$$E_T(N)=\left\|y(x)-y_N(x)\right\|\leq\sum_{r=N+1}^{\infty}\left|a_r\right|$$

Proof: See [22].

Since $T_n(x)$ is a function of $\cos\theta$, $-1\leq T_n(x)\leq 1$.. If we want to change the interval of $T_n(x)$ as $[0,1]$, we can use the transformation $y=2x-1$. Then the Chebyshev polynomials become

$$T_n^*(x)=T_n(y)=T_n(2x-1)$$

which is called the shifted Chebyshev polynomials of the first kind.

Some properties can be written as [22]:

(i)
$$x_i=\frac{1}{2}\left(1+\cos\left(\frac{(2(n-i)+1)\pi}{2(n+1)}\right)\right),\quad i=0,1,\dots,n \tag{6}$$
are roots of $T_{n+1}^*(x)$.

(ii)
$$x^n=2^{-2n+1}{\sum_{k=0}^{n}}{}'\binom{2n}{k}T_{n-k}^*(x),\quad 0\leq x\leq 1 \tag{7}$$
where $\sum'$ denotes a sum whose first term is halved.

Clearly, Theorems 1, 2 and 3 can be converted for the shifted Chebyshev polynomials.

Matrix relations

In this section, the matrix–vector form of the each part of Eq. (1) is introduced by using Eqs. (3) and (6).

Matrix representation of differential part

To obtain the numerical results of Eq. (1), we construct the fundamental matrix–vector relations. These relations help us when we use operational method. Firstly, we suppose that the numerical solution can be written in the shifted first kind Chebyshev series form. The matrix–vector form the approximate solution and its derivatives can be written

$$y_N(x) = \mathbf{T}^*(x)\mathbf{A}, y_N^{(k)}(x) = \mathbf{T}^{*(k)}(x)\mathbf{A}, \quad k = 0, \dots, m \tag{8}$$

where

$$\mathbf{T}^*(x) = [T_0^*(x)\ T_1^*(x) \cdots T_N^*(x)], \quad \mathbf{A} = [a_0\ a_1\ \dots\ a_N]^T$$

From Eq. (6), we get the following matrix relation

$$(\mathbf{Y}(x))^T = \mathbf{D}(\mathbf{T}^*(x))^T \quad \text{and} \quad \mathbf{Y}(x) = \mathbf{T}^*(x)\mathbf{D}^T \tag{9}$$

where

$$\mathbf{Y}(x) = [\ 1\ \ x\ \ \dots\ \ x^N\ ]$$

$$\mathbf{D} = \begin{bmatrix} 2^0\binom{0}{0} & 0 & 0 & 0 & \dots & 0 \\ 2^{-2}\binom{2}{1} & 2^{-1}\binom{2}{0} & 0 & 0 & \dots & 0 \\ 2^{-4}\binom{4}{2} & 2^{-3}\binom{4}{1} & 2^{-3}\binom{4}{0} & 0 & \dots & 0 \\ 2^{-6}\binom{6}{3} & 2^{-5}\binom{6}{2} & 2^{-5}\binom{6}{1} & 2^{-5}\binom{6}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N}\binom{2N}{N} & 2^{-2N+1}\binom{2N}{N-1} & 2^{-2N+1}\binom{2N}{N-2} & 2^{-2n+1}\binom{2N}{N-3} & \dots & 2^{-2N+1}\binom{2N}{0} \end{bmatrix}$$

Since the matrix **D** is invertible, Eq. (9) can be clearly written

$$\mathbf{T}^*(x) = \mathbf{Y}(x)(\mathbf{D}^{-1})^T \tag{10}$$

and

$$(\mathbf{T}^*(x))^{(k)} = \mathbf{Y}^{(k)}(x)(\mathbf{D}^{-1})^T, \quad k = 0, \dots, m. \tag{11}$$

The following relation give us $\mathrm{Y}^{(k)}(x)$ in terms of $\mathrm{Y}(x)$

$$\mathbf{Y}^{(1)}(x) = \mathbf{Y}(x)\mathbf{B}^T \tag{12}$$

$$\begin{aligned} &\mathbf{Y}^{(2)}(x) = \mathbf{Y}^{(1)}(x)\mathbf{B}^T = \mathbf{Y}(x)(\mathbf{B}^T)^2 \\ &\vdots \\ &\mathbf{Y}^{(k)}(x) = \mathbf{Y}^{(k)}(x)\mathbf{B}^T = \mathbf{Y}(x)(\mathbf{B}^T)^k \end{aligned} \tag{13}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}$$

If the obtained the matrix forms Eqs. (11) and (13) are substituted into (8), the approximate solution function $y_N(x) = \sum_{n=0}^{N} a_n T_n^*(x)$ can be transformed into the following matrix form

$$y_N^{(k)}(x) = \mathbf{Y}(x)(\mathbf{B}^{\mathrm{T}})^k(\mathbf{D}^T)^{-1}\mathbf{A}, \quad k = 0, \dots, m \tag{14}$$

Matrix representation of Volterra and Fredholm integral part

In this section, we try to find matrix–vector form Volterra and Fredholm integral part in Eq. (1). For this purpose, suppose that the kernel function $V_m(x, t)$ can be written as:

$$V_m(x, t) = \sum_{r=0}^{N} k_{mr}(x)T_r^*(t). \tag{15}$$

and the matrix form of the $V_m(x, t)$ become

$$V_m(x, t) = \mathbf{K}_m(x)\mathbf{T}^T(t) \tag{16}$$

where

$$\mathbf{K}_m(x) = [\ k_{m0}(x)\ \ k_{m1}(x)\ \ k_{m2}(x)\ \ \cdots\ \ k_{mN}(x)\ ]$$

Using Eqs. (14) and (16), we obtain the following matrix–vector form of the Volterra integral part

$$\left[\sum_{m=0}^{p}\mu_m\int_{c_m}^{x}V_m(x,t)y(t)\mathrm{d}t\right]=\sum_{m=0}^{p}\mu_m\int_{c_m}^{x}\mathbf{K}_m(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t \tag{17}$$

Now, assume that the kernel function $F_n(x,t)$ can be written as:

$$F_n(x,t)=\sum_{r=0}^{N}f_{nr}(x)T_r^*(t) \tag{18}$$

Then the matrix form of the kernel function $F_n(x,t)$ become

$$F_n(x,t)=\mathbf{F}_n(x)\mathbf{T}^T(t) \tag{19}$$

where

$$\mathrm{F}_n(x)=[f_{n0}(x)\ f_{n1}(x)\ f_{n2}(x)\ \cdots\ f_{nN}(x)\ ]$$

Using Eqs. (14) and (19), we obtain the following matrix–vector form of the Fredholm integral part

$$\left[\sum_{n=0}^{q}\lambda_n\int_{a_n}^{b_n}F_n(x,t)y(t)\mathrm{d}t\right]=\sum_{n=0}^{q}\lambda_n\int_{a_n}^{b_n}\mathbf{F}_n(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t \tag{20}$$

Method of solution

In this chapter, the matrix–vector form of Eq. (1) and the operatinal matrix method are assembled to erect the numerical method. Firstly, we have to change the form of $f(x)$ into a matrix–vector form. The matrix form of $f(x)$ can be considered

$$f(x)\approx\mathbf{G}^T\mathbf{Y}(x)(\mathbf{D}^T)^{-1} \tag{21}$$

The obtained matrix–vector forms of differential part, Volterra and Fredholm integral part are put into Eq. (1), we obtain the following matrix–vector equation

$$\begin{aligned}&\sum_{k=0}^{m}\mathbf{P}_k(x)\mathbf{Y}(x)(\mathbf{B}^{\mathrm{T}})^k(\mathbf{D}^T)^{-1}\mathbf{A}\\&+\sum_{m=0}^{p}\mu_m\int_{c_m}^{x}\mathbf{K}_m(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t\\&+\sum_{n=0}^{q}\lambda_n\int_{a_n}^{b_n}\mathbf{F}_n(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t\approx\mathbf{G}^T\mathbf{Y}(x)(\mathbf{D}^T)^{-1}\end{aligned} \tag{22}$$

Thus, the residual function $R_N(x)$ can be gained the following equation:

$$\begin{aligned}R_N(x)\approx&\sum_{k=0}^{m}\mathbf{P}_k(x)\mathbf{Y}(x)(\mathbf{B}^{\mathrm{T}})^k(\mathbf{D}^T)^{-1}\mathbf{A}\\&+\sum_{m=0}^{p}\mu_m\int_{c_m}^{x}\mathbf{K}_m(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t\\&+\sum_{n=0}^{q}\lambda_n\int_{a_n}^{b_n}\mathbf{F}_n(x)\mathbf{D}^{-1}\mathbf{Y}^T(t)\mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}\mathrm{d}t-\mathbf{G}^T\mathbf{Y}(x)(\mathbf{D}^T)^{-1}\end{aligned} \tag{23}$$

Using operational matrix method idea, we gain $(N-m+1)$ linear equations as follows:

$$\left\langle R_N(x),T_n^*(x)\right\rangle_w=\int_0^1 w(x)R_N(x)T_n^*(x)\mathrm{d}x=0,\quad n=0,1,\dots,N-m \tag{24}$$

where $w(x)=\left(x-x^2\right)^{-1/2}$. The m-times initial conditions are obtained by

$$\begin{aligned}&\sum_{k=0}^{m-1}\Big(a_{ik}\mathbf{Y}(a)(\mathbf{B}^{\mathbf{T}})^k(\mathbf{D}^T)^{-1}+b_{ik}\mathbf{Y}(b)(\mathbf{B}^{\mathbf{T}})^k(\mathbf{D}^T)^{-1}\\&\quad+c_{ik}\mathbf{Y}(c)(\mathbf{B}^{\mathbf{T}})^k(\mathbf{D}^T)^{-1}\Big)\mathbf{A}=\alpha_i\end{aligned} \tag{25}$$

Hence, we have $(N+1)$ times linear equations including the unknown coefficients in Eq. (3). If we figure out these linear equations by aid of Maple 13, the approximate solution $y_N(x)$ can be obtained from Eq. (3).

Error estimation and convergence analysis

Now, we will discuss error estimation and convergence analysis.

Theorem: *Let assume that.*

$$y_B(x)=\sum_{r=0}^{\infty}b_rT_r^*(x)\cong\sum_{r=0}^{N}b_rT_r^*(x)+\sum_{r=N+1}^{\infty}b_r$$

(which is the best approximation to $y(x)$) are the shifted Chebyshev polynomials expansion of the exact solution $y(x)\in C^{N+1}$ andis the approximate solution the obtained by proposed method. Then, we have

$$y_N(x)=\sum_{r=0}^{N}a_rT_r^*(x)$$

$$\|y(x)-y_N(x)\|_2\le\frac{1}{2^{2N+1}}\left\|y^{(N+1)}(x)\right\|_\infty+\sqrt{\frac{3\pi}{8}}\|B-A\|_2 \tag{26}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_0 & b_1 & \cdots & b_N \end{bmatrix}.$$

Proof: Firstly, the following inequality is held.

$$\|y(x) - y_N(x)\|_2 \le \|y(x) - y_B(x)\|_2 + \|y_B(x) - y_N(x)\|_2.$$

From Eq. (4), we have the following inequality

$$\begin{aligned} &\|y(x) - y_B(x)\|_2 \\ &= \left(\int_0^1 w(x)\left|y(x) - y_B(x)\right|^2 dx \right)^{1/2} \\ &\le \left(\int_0^1 \left[\frac{1}{2^{2N+1}(N+1)!} \left\|y^{(N+1)}(x)\right\|_\infty \right]^2 dx \right)^{1/2} \\ &= \frac{1}{2^{2N+1}(N+1)!} \left\|y^{(N+1)}(x)\right\|_\infty \end{aligned}$$

and we have

$$\begin{aligned} &\|y_B(x) - y_N(x)\|_2 \\ &= \left(\int_0^1 \left[\sum_{r=0}^{N} (b_r - a_r) T_r^*(x) \right]^2 dx \right)^{1/2} \\ &\le \left(\int_0^1 \left[\sum_{r=0}^{N} (b_r - a_r)^2 \right] \left[\sum_{r=0}^{N} \left|T_r^*(x)\right|^2 \right] dx \right)^{1/2} \\ &= \left[\sum_{r=0}^{N} (b_r - a_r)^2 \right]^{1/2} \left(\sum_{r=0}^{N} \int_0^1 \left|T_r^*(x)\right|^2 dx \right)^{1/2} \\ &= \sqrt{\frac{3\pi}{8}} \|B - A\| . \end{aligned}$$

On the other hand, since the approximate solution $y_N(x)$ is the approximate solution of Eq. (1), then Eq. (1) must be approximately satisfied by the function $y_N(x)$.

$$\left| \sum_{k=0}^{m} P_k(x) y^{(m)}(x) + \sum_{m=0}^{p} \mu_m \int_{c_m}^{x} V_m(x,t) y(t) \mathrm{d}t + \sum_{n=0}^{q} \lambda_n \int_{a_n}^{b_n} F_n(x,t) y(t) \mathrm{d}t = f(x) \right| \approx 0 \tag{27}$$

The following comparison strongly advise by [20]:

$$E_N = \sum_{k=0}^{m} P_k(x) y^{(m)}(x) + \sum_{m=0}^{p} \mu_m \int_{c_m}^{x} V_m(x,t) y(t) \mathrm{d}t + \sum_{n=0}^{q} \lambda_n \int_{a_n}^{b_n} F_n(x,t) y(t) \mathrm{d}t - f(x) \tag{28}$$

which is called error estimation function.

Illustrative examples

In this section, we apply our method some examples to check the accuracy and effectiveness of the method. In examples, some comparisons are dispalyed by below fundamental error types:

1. Absolute error (N_e) is defined by:

   $$N_e(x) = \left|y(x) - y_N(x)\right|, \quad x \in [0,1]$$

   where $y(x)$ are the exact solution and $y_N(x)$ denote the approximate solution.
2. rel$N$ is relative error which is defined by

   $$\mathrm{rel}N = \frac{\left|y(x) - y_N(x)\right|}{|y(x)|}, \quad x \in [0,1]$$

Example 1. Firstly, we apply our method to following the linear FVIDE with piecewise intervals subject to $y(0) = 1$, $y'(0) = -2$, $y''(0) - 2y'(0) = 7$.

$$y'''(x) - y'(x) + \int_0^x xty(t)\mathrm{d}t + \int_{1/4}^{1/2} (1 - t^2x^2) y(t)\mathrm{d}t - \int_{1/2}^{1} (1 - t) y(t)\mathrm{d}t = f(x)$$

where

$$f(x) = (-2 + x + x^2 + x^3)e^x + 0.0861606918 - 0.014742845x^2$$

The exact solution is $y(x) = (1 - x)e^{-x}$. Then, we have

$$P_0(x) = 0, \; P_1(x) = -1, \; P_2(x) = 0, \; P_3(x) = 1$$

$$\lambda_1 = 1,\ F_0(x,t) = xt,\ c_0 = 0$$

$$\mu_1 = 1, \mu_2 = -1,\ V_0(x,t) = 1 - x^2t^2,\ V_1(x,t) = 1 - t,\ a_0 = 1/4,\ b_0 = 1/4,\ a_1 = 1/2,\ b_1 = 1$$

If these values and functions are put into Eq. (24), we have the following approximate solutions for various N values

$$y_5(x) = 1 - 2x + \frac{3}{2}x^2 - 0.6605636x^3 + 0.18906493x^4 - 0.028481763x^5$$

$$y_6(x) = 1 - 2x + \frac{3}{2}x^2 - 0.66617680x^3 + 0.20580958x^4 - 0.0452840x^5 + 0.0055757467x^6$$

$$y_7(x) = 1 - 2x + \frac{3}{2}x^2 - 0.66663491x^3 + 0.2080941625x^4 - 0.0493161690x^5 + 0.008770421x^6 - 0.000912765x^7$$

Comparison of these approximate solutions and exact solution are presented in Table 1. Table 2 gives us the comparison of the relative errors. Figure 1 shows us comparison of absolute errors, Fig. 2 and 3 display error estimation functions and relative error functions, respectively. These figures say that if the N values are increased, the absolute values and relative errors are decreased. Hence, the numerical results are more close with the exact solution.

Example 2. Let us consider the following linear Fredholm integro-differential equation with piecewise intervals [5]

$$y'''(x) = e^x - x - 4\int_0^{1/4} e^{x+t}y(t)\mathrm{d}t + 2\int_0^{1/2} xe^t y(t)\mathrm{d}t - \int_0^1 e^{t-x}y(t)\mathrm{d}t,$$
$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 1$$

If the above numerical algorithm is applied, we have the following numerical solutions

$$y_6(x) = 1 - x + 0.5x^2 - 0.166602262x^3 + 0.0413353094x^4 - 0.00770776517x^5 + 0.00085415183x^6$$

$$y_8(x) = 1 - x + 0.5x^2 - 0.166664113x^3 + 0.0416480144x^4 - 0.00828084359x^5 + 0.001315741562x^6 - 0.0001454724104x^7 + 0.61145088e - 5x^8$$

Also, this problem has been solved by Bernoulli matrix method (BMM) [5]. Table 3 and Fig. 4 show the comparison of Present Method and BMM. From Fig. 4, our numerical results are better than BMM.

Example 3. Let us consider the following linear FVIDE with piecewise intervals.

$$y''(x) - (1-x)y' + y = f(x) + \int_0^{1/2} xty(t)\mathrm{d}t + \int_{1/2}^1 (1-xt)y(t)\mathrm{d}t + \int_0^x (xt^2 - x^2t)y(t)\mathrm{d}t,\ y(0) = 0,\ y'(0) = 0.$$

If we choose $f(x) = 24x^3 - 12x^2 - 10x^4 + 6x^5 - \frac{1}{48}x + \frac{19}{640} + \frac{1}{56}x^9 - \frac{1}{42}x^8$, the exact solution is $x^5 - x^4$. When solving this example by mention method, we get the exact solution for $N = 5$.

Example 4. Let us consider the following equation [33]

$$y'(x) - y(x) = -e^x - e + 2 + \int_0^1 y(t)\mathrm{d}t + \int_0^x y(t)\mathrm{d}t$$

with nonlocal boundary condition

$$y(0) + \int_0^1 y(t)\mathrm{d}t = e$$

The exact solution of this problem is $y(x) = e^x$. In Table 4, we compare our results with the existing method Chebyshev collocation method [33]. The comparison of these results in Fig. 5 and Table 4 shows that our numerical results have a perfect harmony with the exact solution.

Example 5. Let us consider the following Fredholm integro differential equation [7, 34]:

$$y'''(x) = \sin(x) - x - \int_0^{\pi/2} xty'(t)\mathrm{d}t \quad \text{and}$$
$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -1$$

The exact solution is $y(x) = \cos(x)$. The Bernstein projection method (BPM), the variational iteration method (VIM) and the proposed method (PM) are compare in Table 5. It can be observed from Table 5 that PM has less errors compare with BPM and VIM.

Steps of Solutions

In this section, steps of solution have been presented the given numerical method. Maple program is used in this article for run it. Readers can apply the algorithm any computer program.

Algorithm:

(a) Input values and function should be determined N, μ_m, λ_n, $P_k(x)$, $V_m(x,t)$, $F_n(x,t)$, $f(x)$, c_m, a_n, a_{ik}, b_{ik}, c_{ik}, a, b, c and α_i.

(b) Take suitable matrices for A, D, $Y(x)$, B, K_m, F_n,G

(c) Using Eqs.(24)-(25), construct the $R_N(x)$ and $(N-m)$ linear equations from mixed conditions

(d) Solve the obtained linear equations on (c) with conditions

(e) Substituting all coefficients into Eq. (3), this is approximate solution

Table 1 Error values of Ex. 1 for the x value

x	Exact solution	$N_e=5$	$N_e=6$	$N_e=7$
0.1	0.8143536762	0.43818E−5	0.28141E−6	0.13865E−7
0.2	0.6549846024	0.24278E−4	0.11686E−5	0.37154E−7
0.3	0.5185727544	0.54243E−4	0.17378E−5	0.19435E−7
0.4	0.4021920276	0.80309E−4	0.12867E−5	0.28425E−7
0.5	0.3032653298	0.90721E−4	0.27736E−7	0.55326E−7
0.6	0.2195246544	0.81677E−4	0.11841E−5	0.33501E−7
0.7	0.1489755911	0.58648E−4	0.15469E−5	0.11121E−7
0.8	0.0898657928	0.33728E−4	0.94845E−6	0.29990E−7
0.9	0.0406569659	0.19464E−4	0.11238E−6	0.12180E−7
1.0	0.0	0.19555E−4	0.13123E−6	0.11126E−8

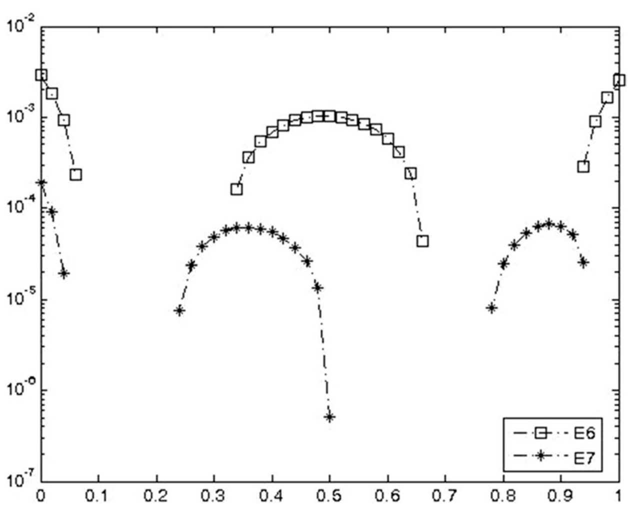

Fig. 2 Comparison of error estimation functions in Ex. 1

Table 2 Some values of relative error

N	rel$N$
5	0.4787E−4
6	0.2765E−5
7	0.3148E−6

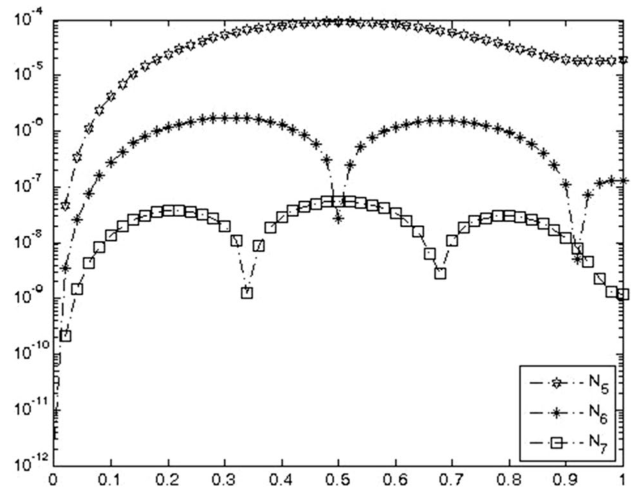

Fig. 1 Comparison of absolute errors in Ex. 1

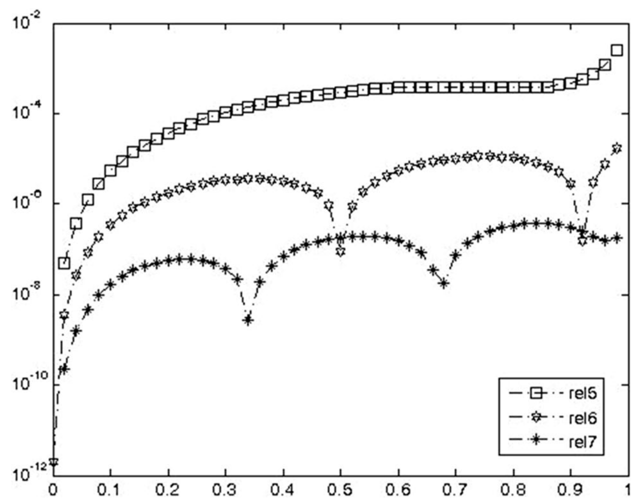

Fig. 3 Comparison of relative error functions in Ex. 1

Conclusion

The operational matrix method is treated as accurate, effective and plain method to gain numerical solutions of the FVIDEs. This method is based on polynomial approximation and basic operational method. By the aid of operational matrices, the all terms of Eq. (1) reduce to a linear algebraic equations. The present method has some considerable advanteges. Since the entry of operational matrices is

Table 3 Numerical comparisons for Ex. 2

x	BMM [5]		Present method	
	$N_e = 6$	$N_e = 8$	$N_e = 6$	$N_e = 8$
0.1	0.31830E−8	0.29967E−10	0.35862E−7	0.114469E−8
0.2	0.22137E−7	0.506946E−9	0.15016E−6	0.332729E−8
0.3	0.37989E−7	0.265453E−8	0.22456E−6	0.247957E−8
0.4	0.12781E−6	0.898136E−8	0.16520E−6	0.103320E−8
0.5	0.11175E−5	0.254436E−7	0.37140E−8	0.353077E−8
0.6	0.45985E−5	0.684357E−7	0.17051E−6	0.271857E−8
0.7	0.14152E−4	0.182284E−6	0.22607E−6	0.14506E−10
0.8	0.36567E−4	0.476192E−6	0.15097E−6	0.142662E−8
0.9	0.83519E−4	0.119350E−5	0.40120E−7	0.486501E−9
1.0	0.17390E−3	0.283062E−5	0.74450E−8	0.266051E−9

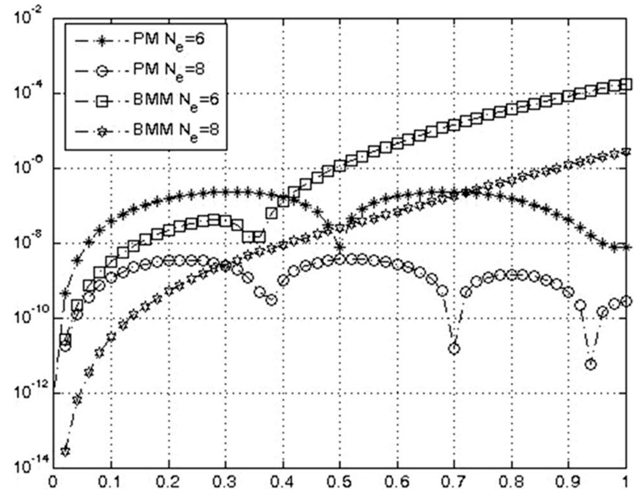

Fig. 4 Comparison of PM and BMM in Ex. 2

Table 4 Numerical comparisons for Ex. 4

x	CCM [33]		Present method	
	$N_e = 5$	$N_e = 6$	$N_e = 5$	$N_e = 6$
0.1	0.329E−3	0.593E−4	0.493E−6	0.119E−7
0.2	0.268E−3	0.388E−4	0.142E−5	0.571E−7
0.3	0.215E−3	0.305E−4	0.114E−5	0.186E−7
0.4	0.170E−3	0.233E−4	0.631E−6	0.654E−7
0.5	0.949E−4	0.170E−5	0.170E−5	0.160E−7
0.6	0.633E−4	0.116E−5	0.908E−6	0.523E−7
0.7	0.364E−4	0.680E−5	0.889E−6	0.390E−7
0.8	0.139E−4	0.246E−5	0.146E−5	0.333E−7
0.9	0.616E−5	0.153E−5	0.326E−6	0.208E−7
1.0	0.302E−4	0.495E−5	0.155E−8	0.140E−9

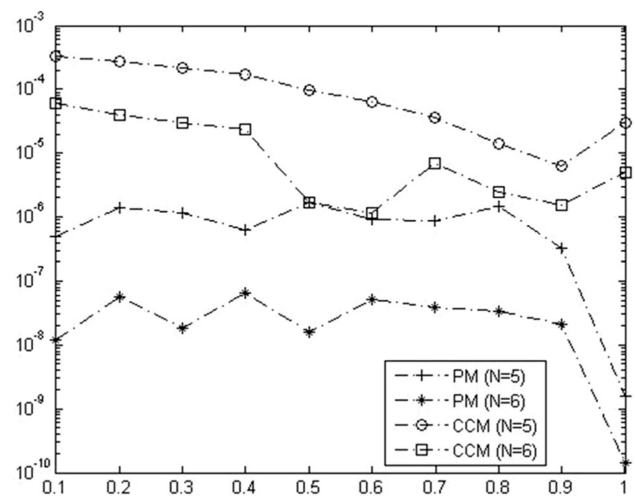

Fig.5 Comparison of PM and CCM

zeroes, the present method has lower operation count and shorter computation time. These advantages bring about less cumulative truncation errors. Also, from Ex.3, if the exact solution is a polynomial, our numerical method give us this polynomial. The proposed method presents us more convenient numerical results than compared methods from Exs. 2, 4 and 5.

Table 5 Comparison of BPM, VIM and PM

x	BPM [7]		VIM [34]		PM	
	$N_e = 6$	$N_e = 12$	$k = 5$	$k = 10$	$N_e = 6$	$N_e = 12$
0.2	6.6E−8	8.2E−15	2.1E−5	6.3E−7	1.03E−7	6.29E−13
0.4	6.6E−7	5.6E−14	3.4E−4	1.0E−5	3.01E−8	1.03E−14
0.6	2.2E−6	1.9E−13	1.7E−3	5.1E−5	6.57E−7	5.24E−13
0.8	5.7E−6	4.7E−13	5.4E−2	1.6E−4	1.80E−6	1.65E−13
1.0	1.2E−5	1.0E−12	1.3E−2	3.9E−4	4.11E−6	4.04E−12

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