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On Generalized ωe^*-closed Sets

Pınar Şaşmaz[1], Murad Özkoç[2,*]

[1] *Muğla Sıtkı Koçman University, Graduate School of Applied and Natural Sciences, Mathematics, 48000 Menteşe-Muğla, Turkey*
[2] *Muğla Sıtkı Koçman University, Faculty of Science, Department of Mathematics, 48000 Menteşe-Muğla, Turkey*

Abstract. The aim of this paper is to introduce and study a new type of generalized closed sets, called generalized ωe^*-closed (briefly, $g\omega e^*$-closed) sets, via ωe^*-closure operator. We examine the fundamental properties of the class of these sets. The notion of $g\omega e^*$-closed set is weaker than the notions of $g\omega\beta$-closed set and ωe^*-closed set in the literature. Also, we define and discuss the notions of generalized ωe^*-continuous and generalized ωe^*-irresolute functions.

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1. Introduction

The notion of the generalized closed set is an important concept in the area of general topology. It was first introduced by Levine [14] in 1970. Since then, many forms of this notion such as $g\alpha$-closed [15], gs-closed [7], gp-closed [16], gb-closed [18], $g\beta$-closed [21], ge-closed [8], πge-closed [9], $g\omega$-closed [5], and generalized $\omega\beta$-closed [4] have been defined and studied by many mathematicians. Moreover, the authors have introduced many new concepts via these new types of sets. They have also investigated some of their fundamental properties and characterizations of these concepts. Furthermore, they have not only discussed their fundamental properties but also put forth the relationships between them and the notions in the literature.

In this study, we define a new concept called generalized ωe^*-closed sets via the ωe^*-closure operator. We examine the relationships among this new concept and some other concepts existing in the literature such as generalized β-closed, generalized e^*-closed, generalized ω-closed, and generalized $\omega\beta$-closed. In addition, by giving the notion of ωe^*-limit point, we prove that the union of two generalized ωe^*-closed sets is a generalized ωe^*-closed set under a special condition. Furthermore, the notions of generalized ωe^*-continuity and generalized ωe^*-irresoluteness have been introduced and finally many basic properties of such functions are obtained.

*Corresponding author.
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Email addresses: pinarsasmaz@posta.mu.edu.tr (P. Şaşmaz), murad.ozkoc@mu.edu.tr (M. Özkoç)

2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted $C(X)$ (resp. $O(X)$ or τ) and the family of all closed (resp. open) sets of X containing a point x of X is denoted by $C(X,x)$ (resp. $O(X,x)$). The family of all neighborhood of a point $x \in X$ is denoted by $\mathcal{N}(x)$.

Definition 1. *A subset A of a space X is called:*

(a) *regular open [20] if $A = int(cl(A))$. The complement of a regular open set is called regular closed. A point $x \in X$ is said to be the δ-cluster point [22] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. If $A = \delta\text{-}cl(A)$, then A is called δ-closed [22], and the complement of a δ-closed set is called δ-open. The set $\{x | (\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by δ-$int(A)$.*

(b) *β-open [1] if $A \subseteq cl(int(cl(A)))$. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by β-$cl(A)$. The union of all β-open sets of X contained in A is called the β-interior of A and is denoted by β-$int(A)$.*

(c) *a-open [10] if $A \subseteq int(cl(\delta\text{-}int(A)))$. The complement of an a-open set is called a-closed [10]. The intersection of all a-closed sets containing A is called the a-closure [10] of A and is denoted by a-$cl(A)$. The union of all a-open sets of X contained in A is called the a-interior [10] of A and is denoted by a-$int(A)$.*

(d) *e*-open [11] if $A \subseteq cl(int(\delta\text{-}cl(A)))$. The complement of an e*-open set is called e*-closed [11]. The intersection of all e*-closed sets containing A is called the e*-closure [11] of A and is denoted by e*-$cl(A)$. The union of all e*-open sets of X contained in A is called the e*-interior [11] of A and is denoted by e*-$int(A)$.*

(e) *ω-open [6] (resp. $\omega\beta$-open [2]) if for every $x \in A$ there exists an open (resp. β-open) set U containing x such that $U \setminus A$ is countable. The complement of an ω-open set (resp. wβ-open set) is said to be ω-closed (resp. $\omega\beta$-closed).*

The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e^*-open, e^*-closed, ω-open, ω-closed, $\omega\beta$-open, $\omega\beta$-closed) subsets of X is denoted by $RO(X)$ (resp. $RC(X)$, $\beta O(X)$, $\beta C(X)$, $aO(X)$, $aC(X)$, $e^*O(X)$, $e^*C(X)$, $\omega O(X)$, $\omega C(X)$, $\omega\beta O(X)$, $\omega\beta C(X)$). The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e^*-open, e^*-closed, ω-open, ω-closed, $\omega\beta$-open, $\omega\beta$-closed) sets of X containing a point x of X is denoted by $RO(X,x)$ (resp. $RC(X,x)$, $\beta O(X,x)$, $\beta C(X,x)$, $aO(X,x)$, $aC(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $\omega O(X,x)$, $\omega C(X,x)$, $\omega\beta O(X,x)$, $\omega\beta C(X,x)$).

Definition 2. *Let A be a subset of a space X. A is said to be ωe^*-open [19] (resp. ωa-open [19]) if for every $x \in A$, there exists an e*-open (resp. a-open) set U containing x such that $U \setminus A$ is countable. The complement of an ωe^*-open (resp. ωa-open) set is called*

ωe^*-closed (resp. ωa-closed). The family of all ωe^*-open (resp. ωe^*-closed, ωa-open, ωa-closed) sets of X will be denoted by $\omega e^*O(X)$ (resp. $\omega e^*C(X)$, $\omega aO(X)$, $\omega aC(X)$). The family of all ωe^*-open (resp. ωe^*-closed, ωa-open, ωa-closed) sets of X containing a point x of X will be denoted by $\omega e^*O(X,x)$ (resp. $\omega e^*C(X,x)$, $\omega aO(X,x)$, $\omega aC(X,x)$).

$$\begin{array}{ccccccc} open & \rightarrow & \beta\text{-}open & \rightarrow & e^*\text{-}open & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \omega\text{-}open & \rightarrow & \omega\beta\text{-}open & \rightarrow & \omega e^*\text{-}open & \leftarrow & \omega a\text{-}open \end{array}$$

Definition 3. *[19] Let A be a subset of a space X. The union of all ωe^*-open subsets of X contained in A is called the ωe^*-interior of A and is denoted by ωe^*-$int(A)$.*

Theorem 1. *[19] Let A be a subset of a space X. Then the following properties hold:*
(a) $\omega e^*\text{-}int(A) \subseteq A$,
(b) $\omega e^*\text{-}int(A) \in \omega e^*O(X)$,
(c) $x \in \omega e^*\text{-}int(A)$ *if and only if there exists* $U \in \omega e^*O(X,x)$ *such that* $U \subseteq A$,
(d) $A \subseteq B \Rightarrow \omega e^*\text{-}int(A) \subseteq \omega e^*\text{-}int(B)$,
(e) $\omega e^*\text{-}int(A) \cup \omega e^*\text{-}int(B) \subseteq \omega e^*\text{-}int(A \cup B)$,
(f) $\omega e^*\text{-}int(A \cap B) \subseteq \omega e^*\text{-}int(A) \cap \omega e^*\text{-}int(B)$,
(g) $A \in \omega e^*O(X)$ *if and only if* $A = \omega e^*\text{-}int(A)$,
(h) $\omega e^*\text{-}int(\omega e^*\text{-}int(A)) = \omega e^*\text{-}int(A)$.

Definition 4. *[19] Let A be a subset of a space X. The intersection of all ωe^*-closed subsets of X containing A is called the ωe^*-closure of A and is denoted by ωe^*-$cl(A)$.*

Theorem 2. *[19] Let A and B be two subsets of a space X. Then the following properties hold:*
(a) $A \subseteq \omega e^*\text{-}cl(A)$,
(b) $\omega e^*\text{-}cl(A) \in \omega e^*C(X)$,
(c) $x \in \omega e^*\text{-}cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \omega e^*O(X,x)$,
(d) $A \subseteq B \Rightarrow \omega e^*\text{-}cl(A) \subseteq \omega e^*\text{-}cl(B)$,
(e) $\omega e^*\text{-}cl(A) \cup \omega e^*\text{-}cl(B) \subseteq \omega e^*\text{-}cl(A \cup B)$,
(f) $\omega e^*\text{-}cl(A \cap B) \subseteq \omega e^*\text{-}cl(A) \cap \omega e^*\text{-}cl(B)$,
(g) $A \in \omega e^*C(X)$ *if and only if* $A = \omega e^*\text{-}cl(A)$,
(h) $\omega e^*\text{-}cl(\omega e^*\text{-}cl(A)) = \omega e^*\text{-}cl(A)$,
(i) $\omega e^*\text{-}cl(X \setminus A) = X \setminus \omega e^*\text{-}int(A)$.

Lemma 1. *[19] Let X be a topological space. Then the following properties hold:*
(a) *The union of any family of we^*-open sets is we^*-open,*
(b) *The intersection of an wa-open set and an we^*-open set is we^*-open.*

Definition 5. *Let A be a subset of a space X. The intersection of all open sets in X containing A is called the kernel [17] of A and is denoted by $ker(A)$.*

Lemma 2. *[17] The followings hold for subsets A and B of a space X.*
(a) $x \in ker(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any* $F \in C(X,x)$,

(b) $A \subseteq ker(A)$,
(c) *If* A *is open in* X, *then* $A = ker(A)$,
(d) *If* $A \subseteq B$, *then* $ker(A) \subseteq ker(B)$.

Definition 6. *A function* $f : X \to Y$ *is called:*
(a) e^**-continuous [11] if* $f^{-1}[V] \in e^*O(X)$ *for each open set* V *of* Y,
(b) β*-continuous [1] if* $f^{-1}[V] \in \omega\beta O(X)$ *for each open set* V *of* Y,
(c) ω*-continuous [13] if* $f^{-1}[V] \in \omega O(X)$ *for each open set* V *of* Y,
(d) $\omega\beta$*-continuous [3] if for each* $x \in X$ *and each open set* V *in* Y *containing* $f(x)$, *there exists an* $\omega\beta$*-open* U *in* X *containing* x *such that* $f[U] \subseteq V$,
(e) ωe^**-continuous [19] at a point* $x \in X$ *if for every open set* V *in* Y *containing* $f(x)$, *there exists an* ωe^**-open set* U *in* X *containing* x *such that* $f[U] \subseteq V$.

Definition 7. *Let* A *be a subset of a space* X. A *is said to be generalized closed [14](briefly, g-closed) (resp. generalized* ω*-closed [5](briefly, g*ω*-closed), generalized* β*-closed [21](briefly, g*β*-closed), generalized* e^**-closed [12](briefly, ge***-closed), generalized* $\omega\beta$*-closed [4](briefly, g*$\omega\beta$*-closed)) if* $cl(A) \subseteq U$ *(resp.* ω*-*$cl(A) \subseteq U$, β*-*$cl(A) \subseteq U$, e^**-*$cl(A) \subseteq U$, $\omega\beta$*-*$cl(A) \subseteq U$*) whenever* $U \in O(X)$ *and* $A \subseteq U$. *The complement of a g-closed (resp. g*ω*-closed [5], g*β*-closed [21], ge***-closed [12], g*$\omega\beta$*-closed [4]) set is called a generalized open (briefly, g-open)(resp. generalized* ω*-open [5](briefly, g*ω*-open), generalized* β*-open [21](briefly, g*β*-open), generalized* e^**-open [12](briefly, ge***-open), generalized* $\omega\beta$*-open [4](briefly, g*$\omega\beta$*-open)). The family of all g-closed (resp. g*ω*-closed [5], g*β*-closed [21], ge***-closed [12], g*$\omega\beta$*-closed [4]) sets of* X *will be denoted by* $gC(X)$ *(resp.* $g\omega C(X)$, $g\beta C(X)$, $ge^*C(X)$, $g\omega\beta C(X)$*). The family of all g-open (resp. g*ω*-open [5], g*β*-open [21], ge***-open[12], g*$\omega\beta$*-open [4]) sets of* X *will be denoted by* $gO(X)$ *(resp.* $g\omega O(X)$, $g\beta O(X)$, $ge^*O(X)$, $g\omega\beta O(X)$*).*

3. Generalized ωe^*-closed Sets

Definition 8. *A subset* A *of a space* X *is called generalized* ωe^**-closed set (briefly, g*ωe^**-closed set) if* ωe^**-*$cl(A) \subseteq U$ *whenever* $U \in O(X)$ *and* $A \subseteq U$. *We denote the family of all generalized* ωe^**-closed subsets of a space* X *by* $g\omega e^*C(X)$.

Proposition 1. *Let* X *be a topological space. Then the followings hold:*
(a) If X *is a countable space, then* $g\omega e^*C(X) = 2^X$,
(b) If $\omega e^*O(X) = \omega e^*C(X)$, *then* $g\omega e^*C(X) = 2^X$.

Proof. (a) Let $A \in 2^X$ and $A \subseteq U \in O(X)$.

$$\left.\begin{array}{r}|X| \le \aleph_0 \Rightarrow \omega e^*C(X) = 2^X \\ A \in 2^X\end{array}\right\} \Rightarrow \; A \in \omega e^*C(X) \Rightarrow \left.\begin{array}{r}\omega e^*\text{-}cl(A) = A \\ A \subseteq U \in O(X)\end{array}\right\} \Rightarrow$$

$\Rightarrow \omega e^*$-$cl(A) \subseteq U$
This means that $A \in g\omega e^*C(X)$. Then we have $2^X \subseteq g\omega e^*C(X)$. On the other hand, we have always $g\omega e^*C(X) \subseteq 2^X$. Therefore $g\omega e^*C(X) = 2^X$.

(b) Let $A \in 2^X$ and $A \subseteq U \in O(X)$.

$$\left.\begin{array}{r} A \subseteq U \in O(X) \\ O(X) \subseteq \omega e^*O(X) = \omega e^*C(X) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) \subseteq \omega e^*\text{-}cl(U) = U$$

This is means that $A \in g\omega e^*C(X)$. Then we have $2^X \subseteq g\omega e^*C(X)$. On the other hand, we have always $g\omega e^*C(X) \subseteq 2^X$. Therefore $g\omega e^*C(X) = 2^X$.

Remark 1. *The following diagram follows immediately from the definitions in which none of the implications is reversible. Also, examples for the other implications are shown in the related papers.*

$$\begin{array}{ccc} closed & \longrightarrow & \omega\text{-}closed \\ \downarrow & & \downarrow \\ g\text{-}closed & \longrightarrow & g\omega\text{-}closed \\ \downarrow & & \downarrow \\ g\beta\text{-}closed & \longrightarrow & g\omega\beta\text{-}closed \\ \downarrow & & \downarrow \\ ge^*\text{-}closed & \longrightarrow & g\omega e^*\text{-}closed \\ \uparrow & & \uparrow \\ e^*\text{-}closed & \longrightarrow & \omega e^*\text{-}closed \end{array}$$

Figure 1: Relationships between some types of closed sets

Example 1. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{a, b\}$. Then A is $g\omega e^*$-closed since X is countable. But A is not ge^*-closed since $A \subseteq \{a, b\} \in O(X)$ but $e^*\text{-}cl(A) = X \nsubseteq \{a, b\}$.*

QUESTION: Is there an example of $g\omega e^*$-closed set which is not ωe^*-closed?

Theorem 3. *Let A be a subset of a space X. If A is $g\omega e^*$-closed, then $\omega e^*\text{-}cl(A) \setminus A$ does not contain any non-empty closed sets.*

Proof. Suppose that $F \in C(X) \setminus \{\emptyset\}$ and $F \subseteq \omega e^*\text{-}cl(A) \setminus A$.

$$\left.\begin{array}{r} (F \in C(X) \setminus \{\emptyset\})(F \subseteq \omega e^*\text{-}cl(A) \setminus A) \Rightarrow A \subseteq X \setminus F \in O(X) \\ A \in g\omega e^*C(X) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow \omega e^*\text{-}cl(A) \subseteq X \setminus F \Rightarrow F \subseteq X \setminus \omega e^*\text{-}cl(A) \\ F \subseteq \omega e^*\text{-}cl(A) \setminus A \end{array}\right\} \Rightarrow F = \emptyset$$

This contradicts with $F \neq \emptyset$.

Theorem 4. *Let A be a $g\omega e^*$-closed subset of a space X. Then A is ωe^*-closed if and only if $\omega e^*\text{-}cl(A) \setminus A$ is closed.*

Proof. $(\Rightarrow)$: It is obvious.
$(\Leftarrow)$: Let $\omega e^*\text{-}cl(A) \setminus A \in C(X)$.

$$\left.\begin{array}{r} A \in g\omega e^*C(X) \overset{\text{Theorem 3}}{\Rightarrow} (\forall F \in C(X))[F \neq \emptyset \Rightarrow F \nsubseteq \omega e^*\text{-}cl(A) \setminus A] \\ \omega e^*\text{-}cl(A) \setminus A \in C(X) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow \omega e^*\text{-}cl(A) \setminus A = \emptyset \Rightarrow \omega e^*\text{-}cl(A) \subseteq A \\ A \subseteq X \Rightarrow A \subseteq \omega e^*\text{-}cl(A) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) = A \Rightarrow A \in \omega e^*C(X).$$

Definition 9. *A space X is called an ωe^*-locally indiscrete space if every open set is ωe^*-closed set.*

Proposition 2. *Let X be a topological space. Then the following are equivalent.*
(a) *X is ωe^*-locally indiscrete;*
(b) *Every subset of X is $g\omega e^*$-closed.*

Proof. $(a) \Rightarrow (b)$: Let $A \subseteq U \in O(X)$.

$$\left.\begin{array}{r} A \subseteq U \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A \subseteq U \in \omega e^*C(X) \Rightarrow \omega e^*\text{-}cl(A) \subseteq \omega e^*\text{-}cl(U) = U.$$

$(b) \Rightarrow (a)$: Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \left.\begin{array}{r} U \in g\omega e^*C(X) \\ U \in O(X) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(U) \subseteq U \Rightarrow U \in \omega e^*C(X).$$

Theorem 5. *Let A be a subset of a space X. If A is both $g\omega e^*$-closed and open, then ωe^*-$cl(A) \setminus A = \emptyset$.*

Proof. Let $A \in O(X) \cap g\omega e^*C(X)$.

$A \in O(X) \cap g\omega e^*C(X) \Rightarrow (A \in O(X))(A \in g\omega e^*C(X))$
$\Rightarrow (A \in O(X))(\forall U \in O(X))(A \subseteq U \Rightarrow \omega e^*\text{-}cl(A) \subseteq U)$
$\Rightarrow \omega e^*\text{-}cl(A) \subseteq A$
$\Rightarrow \omega e^*\text{-}cl(A) \setminus A = \emptyset$.

Theorem 6. *Let A and B be subsets of a space X. If A is $g\omega e^*$-closed and B is any set such that $A \subseteq B \subseteq \omega e^*$-$cl(A)$, then B is $g\omega e^*$-closed.*

Proof. Let $B \subseteq U \in O(X)$.

$$\left.\begin{array}{r} B \subseteq U \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (A \subseteq B \subseteq U)(A \subseteq B \subseteq \omega e^*\text{-}cl(A) \subseteq U)$$

$\Rightarrow \omega e^*\text{-}cl(A) \subseteq \omega e^*\text{-}cl(B) \subseteq \omega e^*\text{-}cl(\omega e^*\text{-}cl(A)) = \omega e^*\text{-}cl(A) \subseteq U$.

Definition 10. *Let A be a subset of a space X. A point $x \in X$ is said to be an ωe^*-limit point of A if for each ωe^*-open set U containing x, we have $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all ωe^*-limit points of A is called the ωe^*-derived set of A and is denoted by $D_{\omega e^*}(A)$.*

Lemma 3. *Let A be a subset of a space X. If $D(A) = D_{\omega e^*}(A)$, then $cl(A) = \omega e^*$-$cl(A)$.*

Proof. It is clear.

Proposition 3. *Let A and B be two subsets of a space X. Then the following properties hold:*
(a) $D_{\omega e^*}(A) \subseteq \omega e^*\text{-}cl(A)$,
(b) $A \subseteq B \Rightarrow D_{\omega e^*}(A) \subseteq D_{\omega e^*}(B)$,
(c) $D_{\omega e^*}(A) \cup D_{\omega e^*}(B) \subseteq D_{\omega e^*}\text{-}cl(A \cup B)$,
(d) $D_{\omega e^*}\text{-}cl(A \cap B) \subseteq D_{\omega e^*}\text{-}cl(A) \cap D_{\omega e^*}\text{-}cl(B)$,
(e) $D_{\omega e^*}(A) \subseteq D(A)$,
(f) $A \in \omega e^*C(X)$ *if and only if* $D_{\omega e^*}(A) \subseteq A$,
(g) $A \cup D_{\omega e^*}(A) \in \omega e^*C(X)$,
(h) $\omega e^*\text{-}cl(A) = A \cup D_{\omega e^*}(A)$.

Proof. The proofs of above results are standard. Hence, they are omitted.

Corollary 1. *Let A be a subset of a space X. If $D(A) \subseteq D_{\omega e^*}(A)$, then for any subsets F and B of X, we have $\omega e^*\text{-}cl(F \cup B) = \omega e^*\text{-}cl(F) \cup \omega e^*\text{-}cl(B)$.*

Proof. It is obvious.

Proposition 4. *Let A and B be subsets of a space X. If A and B are $g\omega e^*$-closed sets such that $D(A) \subseteq D_{\omega e^*}(A)$ and $D(B) \subseteq D_{\omega e^*}(B)$, then $A \cup B$ is $g\omega e^*$-closed.*

Proof. Let $A \cup B \subseteq U \in O(X)$.

$$\left.\begin{array}{r} A \cup B \subseteq U \in O(X) \Rightarrow (A \subseteq U \in O(X))(B \subseteq U \in O(X)) \\ A, B \in g\omega e^*C(X) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\omega e^*\text{-}cl(A) \subseteq U)(\omega e^*\text{-}cl(B) \subseteq U) \\ (D(A) \subseteq D_{\omega e^*}(A))(D(B) \subseteq D_{\omega e^*}(B)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow cl(A \cup B) = cl(A) \cup cl(B) = \omega e^*\text{-}cl(A) \cup \omega e^*\text{-}cl(B) = \omega e^*\text{-}cl(A \cup B) \subseteq U.$$

Proposition 5. *Let A and B be subsets of a space X. Then the following properties hold:*
(a) *If A is open and $g\omega e^*$-closed and B is ωe^*-closed, then $A \cap B$ is $g\omega e^*$-closed,*
(b) *If A is $g\omega e^*$-closed and B is closed, then $A \cap B$ is $g\omega e^*$-closed.*

Proof. (a) Let $A \in O(X) \cap g\omega e^*C(X)$.

$$A \in O(X) \cap g\omega e^*C(X) \overset{\text{Theorem 5}}{\Rightarrow} \omega e^*\text{-}cl(A) \setminus A = \emptyset \Rightarrow \omega e^*\text{-}cl(A) = A$$

$$\left.\begin{array}{r} \Rightarrow A \in \omega e^*C(X) \\ B \in \omega e^*C(X) \end{array}\right\} \Rightarrow A \cap B \in \omega e^*C(X) \subseteq g\omega e^*C(X).$$

(b) Let $A \cap B \subseteq U \in O(X)$.

$$\left.\begin{array}{r} A \cap B \subseteq U \in O(X) \\ B \in C(X) \Rightarrow X \setminus B \in O(X) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow A \subseteq (A \cap B) \cup (X \setminus B) \subseteq U \cup (X \setminus B) \in O(X) \\ A \in g\omega e^*C(X) \end{array}\right\} \Rightarrow$$

$$\begin{aligned}\Rightarrow \omega e^*\text{-}cl(A) \subseteq U \cup (X \setminus B) \Rightarrow \omega e^*\text{-}cl(A \cap B) &\subseteq \omega e^*\text{-}cl(A) \cap \omega e^*\text{-}cl(B)\\ &\subseteq \omega e^*\text{-}cl(A) \cap cl(B)\\ &= \omega e^*\text{-}cl(A) \cap B\\ &\subseteq (U \cup (X \setminus B)) \cap B\\ &= U \cap B\\ &\subseteq U.\end{aligned}$$

Theorem 7. *Let A be a subset of a space X. A is $g\omega e^*$-closed if and only if $cl(\{x\}) \cap A \neq \emptyset$ for every $x \in \omega e^*\text{-}cl(A)$.*

Proof. $(\Rightarrow)$: Suppose that $x \in \omega e^*\text{-}cl(A)$ and $cl(\{x\}) \cap A = \emptyset$.

$$\left.\begin{array}{r} cl(\{x\}) \cap A = \emptyset \Rightarrow A \subseteq X \setminus cl(\{x\}) \in O(X) \\ A \in g\omega e^*C(X) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) \subseteq X \setminus cl(\{x\})$$

$\Rightarrow x \notin \omega e^*\text{-}cl(A)$

This contradicts with $x \in \omega e^*\text{-}cl(A)$.

$(\Leftarrow)$: Let $A \subseteq U \in O(X)$ and $x \in \omega e^*\text{-}cl(A)$.

$$\left.\begin{array}{r} x \in \omega e^*\text{-}cl(A) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A \cap cl(\{x\}) \neq \emptyset \Rightarrow (\exists y \in X)(y \in A \cap cl(\{x\}))$$

$$\left.\begin{array}{r} \Rightarrow (y \in A)(y \in cl(\{x\})) \\ A \subseteq U \in O(X) \end{array}\right\} \Rightarrow (y \in A \subseteq U \in O(X))(y \in cl(\{x\}))$$

$\Rightarrow (U \in O(X, y))(y \in cl(\{x\})) \Rightarrow U \cap \{x\} \neq \emptyset \Rightarrow x \in U.$

Theorem 8. *Let X be a space. For an element $x \in X$, either $\{x\}$ is closed or $X \setminus \{x\}$ is $g\omega e^*$-closed.*

Proof. Suppose that $\{x\} \notin C(X)$.

$$\left.\begin{array}{r} \{x\} \notin C(X) \Rightarrow X \setminus \{x\} \notin O(X) \\ X \setminus \{x\} \subseteq X \in O(X) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(X \setminus \{x\}) \subseteq \omega e^*\text{-}cl(X) = X.$$

Definition 11. *A space X is said to be an ωe^*-$T_{\frac{1}{2}}$ space if for every generalized ωe^*-closed set is ωe^*-closed.*

Example 2. *Any set with indiscrete topology is an example for an ωe^*-$T_{\frac{1}{2}}$ space.*

Theorem 9. *Let X be a space. X is an ωe^*-$T_{\frac{1}{2}}$ space if and only if every singleton is either closed or ωe^*-open.*

Proof. $(\Rightarrow)$: Suppose that $\{x\} \notin C(X)$.

$$\left.\begin{array}{r} \{x\} \notin C(X) \overset{\text{Theorem 8}}{\Rightarrow} X \setminus \{x\} \in g\omega e^*C(X) \\ X \text{ is } \omega e^*\text{-}T_{\frac{1}{2}}\text{space} \end{array}\right\} \Rightarrow X \setminus \{x\} \in \omega e^*C(X)$$

$\Rightarrow \{x\} \in \omega e^*O(X).$

$(\Leftarrow)$: Let $A \in g\omega e^*C(X)$ and $x \in \omega e^*\text{-}cl(A)$.

1st case: Let $\{x\} \in C(X)$ and suppose that $x \notin A$.

$$\left.\begin{array}{r}(\{x\} \in C(X))(x \notin A) \\ x \in \omega e^*\text{-}cl(A)\end{array}\right\} \Rightarrow x \in \omega e^*\text{-}cl(A) \setminus A \Rightarrow \{x\} \subseteq \omega e^*\text{-}cl(A) \setminus A$$

This result contradicts with Theorem 3. Hence, $x \in A$. This means that $\omega e^*\text{-}cl(A) \subseteq A$. Then, we have $A \in \omega e^*$-closed.

2nd case: Let $\{x\} \in \omega e^*O(X)$.

$$\left.\begin{array}{r}x \in \omega e^*\text{-}cl(A) \Rightarrow (\forall U \in \omega e^*O(X,x))(U \cap A \neq \emptyset) \\ \{x\} \in \omega e^*O(X)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\{x\} \in \omega e^*O(X,x))(\{x\} \cap A \neq \emptyset) \Rightarrow x \in A$$

This means that A is ωe^*-closed.

Definition 12. *A space X is said to be an e^*-anti-locally countable if each $U \in e^*O(X) \setminus \{\emptyset\}$ is uncountable.*

Theorem 10. *Let X be a space. If X is e^*-anti-locally countable and ωe^*-$T_{\frac{1}{2}}$ space, then X is T_1 space.*

Proof. Let $x \in X$ and suppose that $\{x\} \notin C(X)$.

$$\left.\begin{array}{r}\{x\} \notin C(X) \overset{\text{Theorem 8}}{\Rightarrow} X \setminus \{x\} \in g\omega e^*C(X) \\ X \text{ is } \omega e^*\text{-}T_{\frac{1}{2}}\end{array}\right\} \Rightarrow X \setminus \{x\} \in \omega e^*C(X)$$

$$\Rightarrow x \in \{x\} \in \omega e^*O(X) \Rightarrow (\exists U \in e^*O(X,x))(|U \setminus \{x\}| \leq \aleph_0)$$

This contradicts the fact that X is e^*-anti-locally countable. Then, $\{x\} \in C(X)$ for all $x \in X$. Namely, X is T_1 space.

Proposition 6. *Let A be a subset of a space X. A is $g\omega e^*$-closed set if and only if $\omega e^*\text{-}cl(A) \subseteq ker(A)$.*

Proof. $(\Rightarrow)$: Let $A \in g\omega e^*C(X)$.

$$\left.\begin{array}{r}A \in g\omega e^*C(X) \Rightarrow (\forall U \in O(X))(A \subseteq U \Rightarrow \omega e^*\text{-}cl(A) \subseteq U) \\ ker(A) := \cap\{U | (A \subseteq U)(U \in O(X))\}\end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) \subseteq ker(A).$$

$(\Leftarrow)$: Let $A \subseteq U \in O(X)$.

$$\left.\begin{array}{r}\left.\begin{array}{r}A \subseteq U \Rightarrow ker(A) \subseteq ker(U) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) \subseteq ker(A) \subseteq ker(U) \\ U \in O(X) \Rightarrow U = ker(U)\end{array}\right\} \Rightarrow$$

$$\Rightarrow \omega e^*\text{-}cl(A) \subseteq U.$$

4. Generalized ωe^*-open Sets and Generalized ωe^*-neighborhoods

Definition 13. *A subset A of a space X is called generalized ωe^*-open if its complement is generalized ωe^*-closed. We denote the family of all generalized ωe^*-open subsets of a space X by $g\omega e^*O(X)$.*

Corollary 2. *Let A be a subset of a space X. A is $g\omega e^*$-open set if and only if $F \subseteq \omega e^*$-$int(A)$, where F is closed set and $F \subseteq A$.*

Proof. $(\Rightarrow)$: Let $A \in g\omega e^*O(X)$ and $F \in C(X)$ such that $F \subseteq A$.

$$\left.\begin{array}{r} A \in g\omega e^*O(X) \Rightarrow X \setminus A \in g\omega e^*C(X) \\ A \supseteq F \in C(X) \Rightarrow X \setminus A \subseteq X \setminus F \in O(X) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \omega e^*\text{-}cl(X \setminus A) = X \setminus \omega e^*\text{-}int(A) \subseteq X \setminus F \Rightarrow F \subseteq \omega e^*\text{-}int(A).$$

$(\Leftarrow)$: Let $X \setminus A \subseteq U \in O(X)$.

$$\left.\begin{array}{r} X \setminus A \subseteq U \in O(X) \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq A) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow X \setminus U \subseteq \omega e^*\text{-}int(A)$$

$$\Rightarrow \omega e^*\text{-}cl(X \setminus A) = X \setminus \omega e^*\text{-}int(A) \subseteq U.$$

Proposition 7. *Let A and B be subsets of a space X. If ωe^*-$int(A) \subseteq B \subseteq A$ and A is $g\omega e^*$-open, then B is $g\omega e^*$-open.*

Proof. It is clear from Theorem 6.

Proposition 8. *Let A be a subset of a space X. If A is $g\omega e^*$-closed, then ωe^*-$cl(A) \setminus A$ is $g\omega e^*$-open.*

Proof. It is clear from Theorem 4.

Remark 2. *Let A be a subset of a space X. Then ωe^*-$int(\omega e^*$-$cl(A) \setminus A) = \emptyset$.*

Proposition 9. *Let A and B be two subsets of a space X. If $A \subseteq B \subseteq X$ and ωe^*-$cl(A) \setminus A$ is $g\omega e^*$-open, then ωe^*-$cl(A) \setminus B$ is $g\omega e^*$-open.*

Proof. Let $F \in C(X)$ such that $F \subseteq \omega e^*$-$cl(A) \setminus B$.

$$\left.\begin{array}{r} A \subseteq B \Rightarrow \omega e^*\text{-}cl(A) \setminus A \subseteq \omega e^*\text{-}cl(A) \setminus B) \\ (F \in C(X))(F \subseteq \omega e^*\text{-}cl(A) \setminus B) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (F \in C(X))(F \subseteq \omega e^*\text{-}cl(A) \setminus A \subseteq \omega e^*\text{-}cl(A) \setminus B) \\ \omega e^*\text{-}cl(A) \setminus A \in g\omega e^*O(X) \end{array}\right\} \overset{\text{Corollary 2}}{\Rightarrow}$$

$$\Rightarrow F \subseteq \omega e^*\text{-}int(\omega e^*\text{-}cl(A) \setminus A) \subseteq \omega e^*\text{-}int(\omega e^*\text{-}cl(A) \setminus B).$$

Proposition 10. *Let A be a subset of a space X. If A is $g\omega e^*$-open, then $U = X$ whenever U is open in X and ωe^*-$int(A) \cup (X \setminus A) \subseteq U$.*

Proof. Let ωe^*-$int(A) \cup (X \setminus A) \subseteq U \in O(X)$.

$$\omega e^*\text{-}int(A) \cup (X \setminus A) \subseteq U \in O(X) \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq X \setminus [\omega e^*\text{-}int(A) \cup (X \setminus A)]$$

$$\left.\begin{array}{r} \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq X \setminus [\omega e^*\text{-}int(A) \cup (X \setminus A)] = \omega e^*\text{-}cl(X \setminus A) \setminus (X \setminus A) \\ A \in g\omega e^*O(X) \Rightarrow X \setminus A \in g\omega e^*C(X) \end{array}\right\} \Rightarrow$$

$$\overset{\text{Theorem 3}}{\Rightarrow} \left.\begin{array}{r} X \setminus U = \emptyset \Rightarrow X \subseteq U \\ U \subseteq X \end{array}\right\} \Rightarrow U = X.$$

Theorem 11. *Let A and B be two subsets of a space X. Then the following properties hold:*
(a) If A is $g\omega e^*$-open and B is ωa-open, then $A \cap B$ is $g\omega e^*$-open,
(b) If B is $g\omega e^*$-open and ωe^*-$int(B) \subseteq A$, then $A \cap B$ is $g\omega e^*$-open.

Proof. (a) Let $F \in C(X)$ such that $F \subseteq A \cap B$.

$$\left.\begin{array}{r}(F \in C(X))(F \subseteq A \cap B) \Rightarrow (F \in C(X))(F \subseteq A \cap B \subseteq A) \\ A \in g\omega e^*O(X)\end{array}\right\} \overset{\text{Corollary 2}}{\Rightarrow}$$

$$\left.\begin{array}{r}\Rightarrow F \subseteq \omega e^*\text{-}int(A) \\ B \in \omega aO(X)\end{array}\right\} \Rightarrow F = F \cap B \subseteq \omega e^*\text{-}int(A) \cap B = \omega e^*\text{-}int(A \cap B).$$

(b) Let $B \in g\omega e^*O(X)$ and ωe^*-$int(B) \subseteq A$.

$$\left.\begin{array}{r}\omega e^*\text{-}int(B) \subseteq A \Rightarrow B \cap \omega e^*\text{-}int(B) \subseteq A \cap B \subseteq B \\ B \in g\omega e^*O(X)\end{array}\right\} \overset{\text{Proposition 7}}{\Rightarrow} A \cap B \in g\omega e^*O(X).$$

Definition 14. *Let X be a space and $x \in X$. A subset N of X is called a $g\omega e^*$-neighborhood of x if there exists a $g\omega e^*$-open set U such that $x \in U \subseteq N$. The set of all $g\omega e^*$-neighborhoods of x is called the $g\omega e^*$-neighborhood system at x, and is denoted by $\mathcal{N}_{g\omega e^*}(x)$.*

Definition 15. *Let X be a space and $A \subseteq X$. A subset N of X is called a $g\omega e^*$-neighborhood of A if there exists a $g\omega e^*$-open set U such that $A \subseteq U \subseteq N$.*

Corollary 3. *Let X be a space and $x \in X$. Every neighborhood N of x is a $g\omega e^*$-neighborhood of x.*

Remark 3. *A $g\omega e^*$-neighborhood N of x in a space X need not be a neighborhood of x as shown by the following example.*

Example 3. *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $A = \{a, c\}$. Since X is countable, $g\omega e^*O(X) = 2^X$. Then, A is a $g\omega e^*$-neighborhood of the point c, since $\{c\}$ is $g\omega e^*$-open set such that $c \in \{c\} \subseteq \{a, c\}$. However, the set $\{a, c\}$ is not a neighborhood of the point c, since there exists no open set U such that $c \in U \subseteq \{a, c\}$.*

Theorem 12. *Let N be a subset of a space X and $x \in X$. If N is $g\omega e^*$-open, then N is a $g\omega e^*$-neighborhood of x.*

Proof. It is clear.

Theorem 13. *Let N and F be two subsets of a space X and $x \in X$. If F is $g\omega e^*$-closed and $x \in X \setminus F$, then there exists a $g\omega e^*$-neighborhood N of x such that $N \cap F = \emptyset$.*

Proof. It is clear.

Theorem 14. *Let N be a subset of a space X and $x \in X$. Then the following properties hold:*
(a) *For all $x \in X$, $\mathcal{N}_{g\omega e^*}(x) \neq \emptyset$,*
(b) *If $N \in \mathcal{N}_{g\omega e^*}(x)$, then $x \in N$,*
(c) *If $N \in \mathcal{N}_{g\omega e^*}(x)$ and $N \subseteq M \subseteq X$, then $M \in \mathcal{N}_{g\omega e^*}(x)$,*
(d) *If $N \in \mathcal{N}_{g\omega e^*}(x)$, then there exists $M \in \mathcal{N}_{g\omega e^*}(x)$ such that $M \subseteq N$ and $N \in \mathcal{N}_{g\omega e^*}(y)$ for every $y \in M$.*

Proof. Straightforward.

Definition 16. *Let A be a subset of a space X. The intersection of all generalized ωe^*-closed (resp. generalized closed [14]) subsets of X containing A is called the generalized ωe^*-closure (resp. generalized closure [14]) of A and is denoted by $g\omega e^*$-$cl(A)$ (resp. g-$cl(A)$).*

The proofs of the following results are standard, hence they are omitted.

Theorem 15. *Let A and B be subsets of a space X and $x \in X$. Then the following properties hold:*
(a) *$x \in g\omega e^*$-$cl(A)$ iff $V \cap A \neq \emptyset$ for every $g\omega e^*$-open set V containing x,*
(b) *$g\omega e^*$-$cl(\emptyset) = \emptyset$ and $g\omega e^*$-$cl(X) = X$,*
(c) *If $A \subseteq B$, then $g\omega e^*$-$cl(A) \subseteq g\omega e^*$-$cl(B)$,*
(d) *$A \subseteq g\omega e^*$-$cl(A) \subseteq \omega e^*$-$cl(A) \subseteq cl(A)$,*
(e) *$A \subseteq g\omega e^*$-$cl(A) \subseteq g$-$cl(A) \subseteq cl(A)$,*
(f) *$g\omega e^*$-$cl(A) \cup g\omega e^*$-$cl(B) \subseteq g\omega e^*$-$cl(A \cup B)$,*
(g) *$g\omega e^*$-$cl(A \cap B) \subseteq g\omega e^*$-$cl(A) \cap g\omega e^*$-$cl(B)$,*
(h) *$A \in g\omega e^*C(X)$ if and only if $A = g\omega e^*$-$cl(A)$,*
(i) *$g\omega e^*$-$cl(A) = g\omega e^*$-$cl(g\omega e^*$-$cl(A))$,*
(j) *$g\omega e^*$-$cl(A) \in g\omega e^*C(X)$.*

Definition 17. *Let X be a topological space.*
(a) *[14] $\tau^* = \{U \subseteq X | cl^*(X \setminus U) = X \setminus U\}$,*
(b) *$\tau^*_{\omega e^*} = \{V \subseteq X | g\omega e^*\text{-}cl(X \setminus V) = X \setminus V\}$.*

Proposition 11. *For a subset A of X, the following properties hold:*
(a) *$\tau \subseteq \omega e^*O(X) \subseteq \tau^*_{\omega e^*}$,*
(b) *$\tau \subseteq gO(X) \subseteq \tau^* \subseteq \tau^*_{\omega e^*}$.*

Theorem 16. *Let X be a topological space. If the family $g\omega e^*O(X)$ is a topology on X, then the family $\tau^*_{\omega e^*}$ is a topology on X.*

Proof. It is obvious that $\emptyset, X \in \tau^*_{\omega e^*}$. Let $A, B \in \tau^*_{\omega e^*}$.

$$\left.\begin{array}{r} A, B \in \tau^*_{\omega e^*} \Rightarrow (g\omega e^*\text{-}cl(X \setminus A) = X \setminus A)(g\omega e^*\text{-}cl(X \setminus B) = X \setminus B) \\ g\omega e^*O(X) \text{ is a topology on } X \end{array}\right\} \Rightarrow$$

$$\Rightarrow g\omega e^*\text{-}cl(X \setminus A) \cup g\omega e^*\text{-}cl(X \setminus B) = (X \setminus A) \cup (X \setminus B)$$

$$\Rightarrow g\omega e^*\text{-}cl((X \setminus A) \cup (X \setminus B)) = g\omega e^*\text{-}cl(X \setminus (A \cap B)) = X \setminus (A \cap B)$$

$\Rightarrow A \cap B \in \tau^*_{\omega e^*}$.

Now, let $\mathcal{A} \subseteq \tau^*_{\omega e^*}$.

$$\left.\begin{array}{r} A \in \mathcal{A} \subseteq \tau^*_{\omega e^*} \Rightarrow g\omega e^*\text{-}cl(X \setminus A) = X \setminus A \Rightarrow X \setminus A \in g\omega e^*C(X) \\ g\omega e^*O(X) \text{ is a topology on } X \end{array}\right\} \Rightarrow$$

$\Rightarrow X \setminus (\cup\mathcal{A}) = \cap_{A\in\mathcal{A}}(X \setminus A) \in g\omega e^*C(X)$
$\Rightarrow g\omega e^*\text{-}cl(\cap_{A\in\mathcal{A}}(X \setminus A)) = \cap_{A\in\mathcal{A}}(X \setminus A) = X \setminus (\cup\mathcal{A})$
$\Rightarrow g\omega e^*\text{-}cl(X \setminus (\cup\mathcal{A})) = X \setminus (\cup\mathcal{A})$
$\Rightarrow \cup\mathcal{A} \in \tau^*_{\omega e^*}$.

Theorem 17. *Let X be a topological space. Then the following properties hold:*
(a) A space X is ωe^*-$T_{\frac{1}{2}}$ if and only if $\tau^*_{\omega e^*} = \omega e^*O(X)$,
(b) Every $g\omega e^*$-closed is closed if and only if $\tau^*_{\omega e^*} = \tau$.

Proof. (a) $(\Rightarrow)$: Let $A \in \tau^*_{\omega e^*}$.

$$\left.\begin{array}{r} A \in \tau^*_{\omega e^*} \Rightarrow X \setminus A = g\omega e^*\text{-}cl(X \setminus A) \Rightarrow X \setminus A \in g\omega e^*C(X) \\ X \text{ is } \omega e^*\text{-}T_{\frac{1}{2}} \end{array}\right\} \Rightarrow$$

$\Rightarrow X \setminus A \in \omega e^*C(X) \Rightarrow A \in \omega e^*O(X)$.

$(\Leftarrow)$: Let $A \in g\omega e^*C(X)$.

$$\left.\begin{array}{r} A \in g\omega e^*C(X) \Rightarrow A = g\omega e^*\text{-}cl(A) \Rightarrow X \setminus A \in \tau^*_{\omega e^*} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow X \setminus A \in \omega e^*O(X)$$

$\Rightarrow A \in \omega e^*C(X)$.

(b) $(\Rightarrow)$: Let $A \in \tau^*_{\omega e^*}$.

$$\left.\begin{array}{r} A \in \tau^*_{\omega e^*} \Rightarrow X \setminus A = g\omega e^*\text{-}cl(X \setminus A) \Rightarrow X \setminus A \in g\omega e^*C(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow X \setminus A \in C(X) \Rightarrow A \in \tau$.

$(\Leftarrow)$: Let $A \in g\omega e^*C(X)$.

$$\left.\begin{array}{r} A \in g\omega e^*C(X) \Rightarrow A = g\omega e^*\text{-}cl(A) \Rightarrow X \setminus A \in \tau^*_{\omega e^*} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow X \setminus A \in \tau$$

$\Rightarrow A \in C(X)$.

5. $g\omega e^*$-continuity, $g\omega e^*$-irresoluteness and $g\omega e^*$-closedness

Definition 18. *A function $f : X \to Y$ is said to be $g\omega e^*$-continuous (resp. $g\omega\beta$-continuous [4]) if $f^{-1}[V]$ is $g\omega e^*$-closed (resp. $g\omega\beta$-closed [4]) in X for every closed set V of Y.*

Corollary 4. *Let $f : X \to Y$ be a function. f is $g\omega e^*$-continuous if and only if the inverse image of every open set in Y is $g\omega e^*$-open in X.*

Remark 4. *Every continuous function is $g\omega e^*$-continuous but the converse need not to be true as shown by the following example.*

Example 4. *Consider the real numbers $\mathbb{R}$ with usual topology and let $Y = \{1, 2\}$ with the topology $\tau = \{\emptyset, Y, \{1\}\}$. Define the function $f : \mathbb{R} \to Y$ by*

$$f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 2 & , \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

Then the function f is $g\omega e^*$-continuous but not continuous since $f^{-1}[\{2\}] = \mathbb{R} \setminus \mathbb{Q}$ is not closed in $\mathbb{R}$.

Remark 5. *Let $f : X \to Y$ be a function. Then the following properties hold:*
(a) If $\tau^*_{\omega e^*} = \tau$ in X, then the notion of continuity and the notion of $g\omega e^*$-continuity coincide.
(b) Every $g\omega e^*$-continuous function defined on ωe^*-$T_{\frac{1}{2}}$ space is ωe^*-continuous.

Remark 6. *The following diagram follows immediately from the definitions in which none of the implications is reversible.*

$$\begin{array}{ccccc} \textit{continuous} & \rightarrow & \omega\beta\textit{-continuous} & \rightarrow & g\omega\beta\textit{-continuous} \\ \downarrow & & \downarrow & & \downarrow \\ e^*\textit{-continuous} & \rightarrow & \omega e^*\textit{-continuous} & \rightarrow & g\omega e^*\textit{-continuous} \end{array}$$

Theorem 18. *Let $f : X \to Y$ be a function. If f is $g\omega e^*$-continuous, then $f[g\omega e^*\text{-}cl(A)] \subseteq cl(f[A])$ for every subset A of X.*

Proof. Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow cl(f[A]) \in C(Y) \\ f \text{ is } g\omega e^*\text{-continuous} \end{array}\right\} \Rightarrow f^{-1}[cl(f[A])] \in g\omega e^*C(X)$$

$$\left.\begin{array}{r} \Rightarrow g\omega e^*\text{-}cl(f^{-1}[cl(f[A])]) = f^{-1}[cl(f[A])] \\ A \subseteq f^{-1}[f[A]] \subseteq f^{-1}[cl(f[A])] \Rightarrow g\omega e^*\text{-}cl(A) \subseteq g\omega e^*\text{-}cl(f^{-1}[cl(f[A])]) \end{array}\right\} \Rightarrow$$

$\Rightarrow g\omega e^*\text{-}cl(A) \subseteq f^{-1}[cl(f[A])]$
$\Rightarrow f[g\omega e^*\text{-}cl(A)] \subseteq cl(f[A])$.

Theorem 19. *Let $f : X \to Y$ be a function. If for each point $x \in X$ and each open set V containing $f(x)$ there exists a $g\omega e^*$-open set U containing x such that $f[U] \subseteq V$, then $f[g\omega e^*\text{-}cl(A)] \subseteq cl(f[A])$ for every subset A of X.*

Proof. Let $y \in f[g\omega e^*\text{-}cl(A)]$.
$y \in f[g\omega e^*\text{-}cl(A)] \Rightarrow (\exists x \in g\omega e^*\text{-}cl(A))(f(x) = y)$

$$\left.\begin{array}{r} \Rightarrow (\forall U \in g\omega e^*O(X, x))(U \cap A \neq \emptyset)(f(x) = y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\forall V \in O(Y, f(x)))(U \in g\omega e^*O(X, x))(\emptyset \neq f[U \cap A] \subseteq f[U] \cap f[A] \subseteq V \cap f[A])$
$\Rightarrow y = f(x) \in cl(f[A])$.

Theorem 20. *Let $f : X \to Y$ be a function. Then the following statements are equivalent:*
(a) $f[g\omega e^*\text{-}cl(A)] \subseteq cl(f[A])$ for every subset A of X;
(b) If $\tau^*_{\omega e^*}$ is a topology on X, then $f : (X, \tau^*_{\omega e^*}) \to (Y, \sigma)$ is continuous.

Proof. $(a) \Rightarrow (b)$: Let $A \in C(Y)$.

$$\left.\begin{array}{r} A \in C(Y) \Rightarrow f^{-1}[A] \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[g\omega e^*\text{-}cl(f^{-1}[A])] \subseteq cl(f[f^{-1}[A]]) \subseteq cl(A) = A$$

$$\left.\begin{array}{r} \Rightarrow g\omega e^*\text{-}cl(f^{-1}[A]) \subseteq f^{-1}[A] \\ f^{-1}[A] \subseteq g\omega e^*\text{-}cl(f^{-1}[A]) \end{array}\right\} \Rightarrow f^{-1}[A] = g\omega e^*\text{-}cl(f^{-1}[A])$$

$$\Rightarrow X \setminus f^{-1}[A] \in \tau^*_{\omega e^*} \Rightarrow f^{-1}[A] \in C(X, \tau^*_{\omega e^*}).$$

$(b) \Rightarrow (a)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow cl(f[A]) \in C(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow X \setminus f^{-1}[cl(f[A])] \in \tau^*_{\omega e^*}$$

$$\Rightarrow f^{-1}[cl(f[A])] \in C(X, \tau^*_{\omega e^*})$$
$$\Rightarrow g\omega e^*\text{-}cl(A) \subseteq g\omega e^*\text{-}cl(f^{-1}[cl(f[A])]) = f^{-1}[cl(f[A])]$$
$$\Rightarrow f[g\omega e^*\text{-}cl(A)] \subseteq cl(f[A]).$$

Definition 19. *A function $f : X \to Y$ is said to be pre-ωe^*-closed if $f[F]$ is ωe^*-closed in Y for every ωe^*-closed set F of X.*

Definition 20. *A function $f : X \to Y$ is said to be pre-$g\omega e^*$-closed if $f[U]$ is $g\omega e^*$-closed in Y for every $g\omega e^*$-closed set U of X.*

Theorem 21. *Let $f : X \to Y$ be a function. If f is continuous and pre-ωe^*-closed, then f is pre-$g\omega e^*$-open.*

Proof. Let $A \in g\omega e^* C(X)$ and $f[A] \subseteq U \in O(Y)$.

$$\left.\begin{array}{r} (A \in g\omega e^* C(X))(f[A] \subseteq U \in O(Y)) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (A \subseteq f^{-1}[U] \in O(X))(\omega e^*\text{-}cl(A) \subseteq f^{-1}[U]) \Rightarrow f[\omega e^*\text{-}cl(A)] \subseteq U \\ f \text{ is pre-}\omega e^*\text{-closed function} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \omega e^*\text{-}cl(f[A]) \subseteq \omega e^*\text{-}cl(f[\omega e^*\text{-}cl(A)]) = f[\omega e^*\text{-}cl(A)] \subseteq U.$$

Definition 21. *A function $f : X \to Y$ is said to be $g\omega e^*$-irresolute if $f^{-1}[V]$ is $g\omega e^*$-closed in X for every $g\omega e^*$-closed set V of Y.*

Corollary 5. *Let $f : X \to Y$ be a function. f is $g\omega e^*$-irresolute if $f^{-1}[V]$ is $g\omega e^*$-open in X for every $g\omega e^*$-open set V of Y.*

Proposition 12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is $g\omega e^*$-continuous and $\sigma^*_{\omega e^*} = \sigma$ holds, then f is $g\omega e^*$-irresolute.*

The proof follows from Remark 5.

Theorem 22. *Let $f : X \to Y$ be a function. If f is an ωe^*-irresolute open bijection, then f is $g\omega e^*$-irresolute.*

Proof. Let $F \in g\omega e^*C(Y)$ and $f^{-1}[F] \subseteq U \in O(X)$.

$$\left.\begin{array}{r} f^{-1}[F] \subseteq U \in O(X) \\ f \text{ is open bijection} \end{array}\right\} \Rightarrow \left.\begin{array}{r} F \subseteq f[U] \in O(Y) \\ F \in g\omega e^*C(Y) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(F) \subseteq f[U]$$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[\omega e^*\text{-}cl(F)] \subseteq U \\ f \text{ is } \omega e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[F]) \subseteq \omega e^*\text{-}cl(f^{-1}[\omega e^*\text{-}cl(F)]) = f^{-1}[\omega e^*\text{-}cl(F)] \subseteq U.$$

Theorem 23. *Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then the following properties hold:*
(a) If g is continuous and f is $g\omega e^*$-continuous, then $g \circ f$ is $g\omega e^*$-continuous,
(b) If g is $g\omega e^*$-irresolute and f is $g\omega e^*$-irresolute, then $g \circ f$ is $g\omega e^*$-irresolute,
(c) If g is $g\omega e^*$-continuous and f is $g\omega e^*$-irresolute, then $g \circ f$ is $g\omega e^*$-continuous,
(d) If g is $g\omega e^*$-continuous and f is ωe^*-irresolute and Y is ωe^*-$T_{\frac{1}{2}}$ space, then $g \circ f$ is ωe^*-continuous,
(e) If g and f are $g\omega e^*$-continuous and $\sigma^*_{\omega e^*} = \sigma$, then $g \circ f$ is ωe^*-continuous.

Proof. Straightforward.

Theorem 24. *Let $f : X \to Y$ be a function. Then the following properties hold:*
(a) If f is $g\omega e^*$-irresolute and X is ωe^*-$T_{\frac{1}{2}}$ space, then f is ωe^*-irresolute,
(b) If f is $g\omega e^*$-continuous and X is ωe^*-$T_{\frac{1}{2}}$ space, then f is ωe^*-continuous.

Proof. Straightforward.

Theorem 25. *Let $f : X \to Y$ be a pre-ωe^*-closed and $g\omega e^*$-irresolute surjection. If X is ωe^*-$T_{\frac{1}{2}}$ space, then Y is ωe^*-$T_{\frac{1}{2}}$ space.*

Proof. Let $F \in g\omega e^*C(Y)$.

$$\left.\begin{array}{r} F \in g\omega e^*C(Y) \\ f \text{ is } g\omega e^*\text{-irresolute} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[F] \in g\omega e^*C(X) \\ X \text{ is } \omega e^*\text{-}T_{\frac{1}{2}} \end{array}\right\} \Rightarrow f^{-1}[F] \in \omega e^*C(X)$$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[F] \in \omega e^*C(X) \\ f \text{ is pre-}\omega e^*\text{-closed surjection} \end{array}\right\} \Rightarrow f[f^{-1}[F]] = F \in \omega e^*C(Y).$$

Definition 22. *A function $f : X \to Y$ is said to be $g^*\omega e^*$-continuous if $f^{-1}[V]$ is $g\omega e^*$-closed in X for every ωe^*-closed set V of Y.*

Remark 7. *Recall that every $g\omega e^*$-irresolute function is $g^*\omega e^*$-continuous function and every $g^*\omega e^*$-continuous function is $g\omega e^*$-continuous function.*

Proposition 13. *Let $f : X \to Y$ be a function. If f is an open bijection and $g^*\omega e^*$-continuous, then f is $g\omega e^*$-irresolute.*

Proof. Let $A \in g\omega e^*C(Y)$ and $f^{-1}[A] \subseteq U \in O(X)$.

$$\left.\begin{array}{r} f^{-1}[A] \subseteq U \in O(X) \\ f \text{ is open bijection} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f[f^{-1}[A]] = A \subseteq f[U] \in O(Y) \\ A \in g\omega e^*C(Y) \end{array}\right\} \Rightarrow \omega e^*\text{-}cl(A) \subseteq f[U]$$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[\omega e^*\text{-}cl(A)] \subseteq U \\ f \text{ is } g^*\omega e^*\text{-continuous} \end{array}\right\} \Rightarrow f^{-1}[\omega e^*\text{-}cl(A)] \in g\omega e^*C(X)$$

$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[A]) \subseteq \omega e^*\text{-}cl(f^{-1}[\omega e^*\text{-}cl(A)]) \subseteq U.$$

Proposition 14. *Let $f : X \to Y$ be a pre-ωe^*-closed and $g^*\omega e^*$-continuous bijection open function. If X is ωe^*-$T_{\frac{1}{2}}$ space, then Y is ωe^*-$T_{\frac{1}{2}}$ space.*

Proof. Let $A \in g\omega e^*C(Y)$.

$$\left.\begin{array}{r} A \in g\omega e^*C(Y) \\ f \text{ is } g^*\omega e^*\text{-continuous open bijection} \end{array}\right\} \overset{\text{Proposition 13}}{\Rightarrow} \left.\begin{array}{r} f^{-1}[A] \in g\omega e^*C(X) \\ X \text{ is } \omega e^*\text{-}T_{\frac{1}{2}} \text{ space} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[A] \in \omega e^*C(X) \\ f \text{ is pre-}\omega e^*\text{-closed bijection} \end{array}\right\} \Rightarrow f[f^{-1}[A]] = A \in \omega e^*C(Y).$$

Definition 23. *A function $f : X \to Y$ is said to be $g\omega e^*$-closed if $f[F]$ is $g\omega e^*$-closed in Y for every closed set F of X.*

Remark 8. *Every closed function is $g\omega e^*$-closed function but not conversely.*

Example 5. *Let $X = \{1, 2\}$ with the topologies $\tau = \{X, \emptyset, \{1\}\}$ and $\sigma = \{X, \emptyset, \{2\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is $g\omega e^*$-closed but not closed since $f[\{2\}] = \{2\}$ is not closed in X.*

Theorem 26. *Let $f : X \to Y$ be a function. Then, f is $g\omega e^*$-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}[S]$, there exists a $g\omega e^*$-open set V of Y such that $S \subseteq V$ and $f^{-1}[V] \subseteq U$.*

Proof. $(\Rightarrow)$: Let $S \subseteq Y$ and $f^{-1}[S] \subseteq U \in O(X)$.

$$\left.\begin{array}{r} f^{-1}[S] \subseteq U \in O(X) \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq X \setminus f^{-1}[S]) \\ f \text{ is } g\omega e^*\text{-closed} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[X \setminus U] \in g\omega e^*C(Y))(f[X \setminus U] \subseteq f[X \setminus f^{-1}[S]] = f[f^{-1}[Y \setminus S]] \subseteq Y \setminus S)$$

$$\left.\begin{array}{r} \Rightarrow (Y \setminus f[X \setminus U] \in g\omega e^*O(Y))(S \subseteq Y \setminus f[X \setminus U]) \\ V := Y \setminus f[X \setminus U] \end{array}\right\} \Rightarrow$$

$$\Rightarrow (V \in g\omega e^*O(Y))(S \subseteq V)(f^{-1}[V] \subseteq U).$$

$(\Leftarrow)$: Let $F \in C(X)$.

$$\left.\begin{array}{r} F \in C(X) \Rightarrow f^{-1}[Y \setminus f[F]] \subseteq X \setminus F \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in g\omega e^*O(Y))(Y \setminus f[F] \subseteq V)(f^{-1}[V] \subseteq X \setminus F)$
$\Rightarrow (\exists V \in g\omega e^*O(Y))(Y \setminus V \subseteq f[F] \subseteq f[X \setminus f^{-1}[V]] \subseteq Y \setminus V)$
$\Rightarrow (Y \setminus V \in g\omega e^*C(Y))(Y \setminus V = f[F])$
$\Rightarrow f[F] \in g\omega e^*C(Y)$.

Theorem 27. *Let $f : X \to Y$ be a function. If f is $g\omega e^*$-closed, then $g\omega e^*$-$cl(f[A]) \subseteq f[cl(A)]$ for every subset A of X.*

Proof. Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow cl(A) \in C(X) \\ f \text{ is } g\omega e^*\text{-closed} \end{array}\right\} \Rightarrow f[cl(A)] \in g\omega e^*C(Y)$$

$\Rightarrow g\omega e^*$-$cl(f[A]) \subseteq g\omega e^*$-$cl(f[cl(A)]) = f[cl(A)]$.

Theorem 28. *Let $f : X \to Y$ be a function. If f is continuous, $g\omega e^*$-closed and A is a g-closed subset of X, then $f[A]$ is $g\omega e^*$-closed.*

Proof. Let $f[A] \subseteq U \in O(Y)$.

$$\left.\begin{array}{r} f[A] \subseteq U \in O(Y) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} A \subseteq f^{-1}[f[A]] \subseteq f^{-1}[U] \in O(X) \\ A \in gC(X) \end{array}\right\} \Rightarrow cl(A) \subseteq f^{-1}[U]$$

$$\left.\begin{array}{r} \Rightarrow (cl(A) \in C(X))(f[A] \subseteq f[cl(A)] \subseteq f[f^{-1}[U]] \subseteq U) \\ f \text{ is } g\omega e^*\text{-closed} \end{array}\right\} \Rightarrow$$

$\Rightarrow (f[cl(A)] \in g\omega e^*C(Y))(f[A] \subseteq f[cl(A)] \subseteq U)$
$\Rightarrow \omega e^*$-$cl(f[A]) \subseteq \omega e^*$-$cl(f[cl(A)]) \subseteq U$.

Theorem 29. *Let $f : X \to Y$ be an open bijection. If f is $g^*\omega e^*$-continuous, then f is $g\omega e^*$-irresolute.*

Proof. Let $V \in g\omega e^*C(Y)$ and $f^{-1}[V] \subseteq U \in O(X)$.

$$\left.\begin{array}{r} (V \in g\omega e^*C(Y))(f^{-1}[V] \subseteq U \in O(X)) \\ f \text{ is open bijection} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (f[f^{-1}[V]] = V \subseteq f[U] \in O(Y))(\omega e^*\text{-}cl(V) \subseteq f[U]) \\ f \text{ is } g^*\omega e^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow (f^{-1}[\omega e^*$-$cl(V)] \in g\omega e^*C(X))(f^{-1}[\omega e^*$-$cl(V)] \subseteq U)$
$\Rightarrow \omega e^*$-$cl(f^{-1}[V]) \subseteq \omega e^*$-$cl(f^{-1}[\omega e^*$-$cl(V)]) \subseteq U$.

Theorem 30. *Let $f : X \to Y$ be a function. If f is a continuous pre-ωe^*-closed bijection, then the inverse function of f is $g\omega e^*$-irresolute.*

Proof. Let $A \in g\omega e^*C(X)$ and $(f^{-1})^{-1}[A] = f[A] \subseteq U \in O(Y)$.

$$\left.\begin{array}{r} f[A] \subseteq U \in O(Y) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} A \subseteq f^{-1}[U] \in O(X) \\ A \in g\omega e^*C(X) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow \omega e^*\text{-}cl(A) \subseteq f^{-1}[U] \\ f \text{ is a pre-}\omega e^*\text{-closed bijection}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f[\omega e^*\text{-}cl(A)] \in \omega e^*C(Y))(f[A] \subseteq f[\omega e^*\text{-}cl(A)] \subseteq f[f^{-1}[U]] = U)$$
$$\Rightarrow \omega e^*\text{-}cl(f[A]) \subseteq \omega e^*\text{-}cl(f[\omega e^*\text{-}cl(A)]) = f[\omega e^*\text{-}cl(A)] \subseteq U.$$

Theorem 31. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is a continuous surjection and $g \circ f$ is $g\omega e^*$-closed, then g is $g\omega e^*$-closed.*

Proof. Let $V \in C(Y)$.

$$\left.\begin{array}{r}V \in C(Y) \\ f \text{ is continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}f^{-1}[V] \in C(X) \\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow g[f[f^{-1}[V]]] = (g \circ f)[f^{-1}[V]] = g[V] \\ g \circ f \text{ is } g\omega e^*\text{-closed}\end{array}\right\} \Rightarrow g[V] \in g\omega e^*C(X).$$

Theorem 32. *Let $f : X \to Y$ be a function. If f is $g\omega e^*$-closed continuous and X is normal, then Y is ωe^*-normal.*

Proof. Let $A, B \in C(Y)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}(A, B \in C(Y))(A \cap B = \emptyset) \\ f \text{ is continuous}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (f^{-1}[A], f^{-1}[B] \in C(X))(f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] = f^{-1}[\emptyset] = \emptyset) \\ X \text{ is normal}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists U \in O(X, f^{-1}[A]))(\exists V \in O(X, f^{-1}[B]))(U \cap V = \emptyset) \\ f \text{ is } g\omega e^*\text{-closed}\end{array}\right\} \overset{\text{Theorem 26}}{\Rightarrow}$$
$$\Rightarrow (\exists G, H \in g\omega e^*O(Y))(A \subseteq G)(B \subseteq H)(f^{-1}[G] \subseteq U)(f^{-1}[H] \subseteq V)(U \cap V = \emptyset)$$
$$\left.\begin{array}{r}\Rightarrow (A \subseteq \omega e^*\text{-}int(G))(B \subseteq \omega e^*\text{-}int(H))(A \cap B \subseteq G \cap H)(f^{-1}[G] \cap f^{-1}[H] = \emptyset) \\ (U' := \omega e^*\text{-}int(G))(V' := \omega e^*\text{-}int(H))\end{array}\right\} \Rightarrow$$
$$\Rightarrow (U' \in \omega e^*O(Y, A))(V' \in \omega e^*O(Y, B))(U' \cap V' = \emptyset).$$

Theorem 33. *Let $f : X \to Y$ be a function. If f is a $g\omega e^*$-closed continuous surjection, ωe^*-open and X is regular, then Y is ωe^*-regular.*

Proof. Let $y \in Y$ and $U \in O(Y, y)$.

$$\left.\begin{array}{r}(y \in Y)(U \in O(Y, y)) \\ f \text{ is continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists x \in X)(y = f(x))(f^{-1}[U] \in O(X, x)) \\ X \text{ is regular}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists V \in O(X, x))(V \subseteq cl(V) \subseteq f^{-1}[U]) \\ f \text{ is } g\omega e^*\text{-closed surjection}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists V \in O(X, x))(f[cl(V)] \in g\omega e^*C(X))(y \in f[V] \subseteq f[cl(V)] \subseteq U) \\ f \text{ is } \omega e^*\text{-open}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f[V] \in \omega e^*O(Y, y))(\omega e^*\text{-}cl(f[V]) \subseteq \omega e^*\text{-}cl(f[cl(V)]) \subseteq U).$$

Conclusion

Many forms of generalized closed sets which are first defined by Levine [14] have been studied by many authors in recent years. This paper is concerned with the notion of generalized ωe^*-closed sets which are defined by utilizing the concept of ωe^*-open set. We have seen that this concept is weaker than many generalized closed set forms in the literature as will be seen in Figure 1. In addition, we gave some examples related to the concept but we could not find an example generalized ωe^*-closed set which is not ωe^*-closed. We believe that this study will help researchers to upgrade and support further studies related to compactness and connectedness etc. Also, the objects considered in the article may find an application in the area of both pure and applied sciences such as computational topology and digital topology.

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