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Financial model with chaotic analysis

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ABSTRACT

Recently, it was proposed to use a brand-new set of nonlinear ordinary differential equations. The system aims to represent chaotic financial activities. Such a system was taken into consideration for various analyses in this paper. For each of the three axes, we first evaluated the nullcline points and then gave the formula for the associated Poincare mapping. With various differential operators, we have analyzed the existence and uniqueness of systems of solutions. We have numerically solved the model using the well-known Nystrom in the case of the classical model and the Midpoint in the case of fractional derivatives.

Introduction

In the fields of science, technology, and engineering, nonlinear differential equations are potent mathematical instruments for simulating complicated systems. Since most systems are inherently nonlinear, engineers, biologists, physicists, mathematicians, and many other scientists are interested in nonlinear problems. In contrast to considerably simpler linear systems, which describe changes in variables over time, nonlinear dynamical systems might appear chaotic, unexpected, or paradoxical. They have been shown to be particularly suitable for modeling financial processes. In general, there are two distinct areas of finance that need for advanced quantitative methods: risk and portfolio management and the pricing of derivatives. Financial engineering and computational finance share many similarities with mathematical finance. The latter emphasizes modeling and applications, frequently with the aid of stochastic asset models, whereas the former concentrates on developing tools for putting the models into practice in addition to analysis. Related to this is quantitative investing, which uses numerical and statistical models rather than traditional fundamental analysis to manage portfolios. Numerous academics with various backgrounds in engineering, science, and technology have shown interest in this study. According to the available literature, the PhD dissertation of French mathematician Louis Bachelier, which was approved in 1900, is regarded as the first academic study of mathematical finance. However, after the development of option pricing theory by Fischer Black, Myron Scholes, and Robert Merton, the field of mathematical finance began to take shape in the 1970s [1–4]. Other studies that predict the dynamical behaviors of these processes as functions of time have been conducted in recent years [5–7]. Numerous researchers have used cutting-edge methods like fractional calculus to more accurately model the nonlocal processes that these financial problems illustrate. Some of these suggested mathematical models have been found to exhibit chaotic behavior, which is in complete agreement with what is seen in real-world scenarios. In [8], they have proposed a nonlinear system of ordinary differential equations and gave some analysis therein. We will offer several analyses and broaden the concept to include the framework of nonlocal operators in this study.

Definitions of derivatives

In this section, we present some definitions and theorems that will be used in the paper [9–11].

Definition 1. Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to R$, according to Caputo, the fractional derivative of a continuous and differentiable function f is given as :

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} \frac{d}{dx} f(x)dx, \quad 0 < \alpha \le 1. \tag{1}$$

Definition 2. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$, according to Riemann–Liouville, the fractional integral that is considered as anti-fractional derivative of a function f is :

$$ {}_0^{RL} I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \tag{2}$$

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Definition 3. Let $f \in H^1(a,b)$, $b > a$, $0 < \alpha < 1$ then, the new Caputo–Fabrizio derivative of fractional derivative is defined as :

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{1}{1-\alpha}\int_a^t f'(x)\exp\left[-\alpha\frac{(t-x)}{1-\alpha}\right]dx. \tag{3}$$

and also if the function does not belongs to $H^1(a,b)$ then, the derivative can be reformulated as

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{\alpha}{1-\alpha}\int_a^t (f(t)-f(x))\exp\left[-\alpha\frac{(t-x)}{1-\alpha}\right]dx. \tag{4}$$

Theorem 1. *Let $0 < \alpha < 1$ then the following time fractional ordinary differential equation*

$$ {}_{0}^{CF}D_t^{\alpha} f(t) = u(t), \tag{5}$$

has a unique solution with taking the inverse Laplace transform and using the convolution theorem below [12]:

$$ f(t) = (1-\alpha)u(t) + \alpha\int_0^t u(s)ds,\ t \geq 0. \tag{6}$$

Definition 4. Let $f(t)$ be continuous, $g(t)$ be a non-constant increasing positive function. And also taking $K(t)$ as kernel with singular or non-singular versions. For $0 < \alpha \leq 1$, a fractional global derivative of Caputo sense is defined by

$$ {}_{0}^{C}D_g^{\alpha} f(t) = D_g f(t) * K(t). \tag{7}$$

Also with Riemann–Liouville version, we have

$$ {}_{0}^{RL}D_g^{\alpha} f(t) = D_g\left(f(t) * K(t)\right), \tag{8}$$

where $*$ means the convolution operator.

Theorem 2. *Let $0 < \alpha \leq 1$ then the following time fractional ordinary differential equation*

$$ {}_{0}^{C}D_t^{\alpha} f(t) = u(t), \tag{9}$$

has a unique solution with taking the inverse Laplace transform and using the convolution theorem below:

$$ f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,\ t \geq 0. \tag{10}$$

Financial model

Financial modeling is the construction of an abstract representation (a model) of a real financial situation. In other words, financial modeling is about translating a set of hypotheses about the behavior of markets or intermediaries into numerical predictions. In this section, we consider the model formulation of the financial system that has been considered in [8,13] recently. In [8], they proposed a model using differential equations to investigate the behavior of a financial system containing interest rates, investment demand, and a price index. The model captures the interaction between a variety of financial factors. In the following, based on the model presented in [8], we work for the model to consider the interplay between the interest rate $x(t)$, the investment demand $y(t)$ and the price index $z(t)$. Model is given by

$$\begin{aligned} \frac{dx}{dt} &= gz + (y-a)x, \\ \frac{dy}{dt} &= -by^3 - sx^2 + r, \\ \frac{dz}{dt} &= -cz - \beta x - py \end{aligned} \tag{11}$$

where a, b, c, p, r, s, and β are constants. System (11) assumes that the rate of change of interest rate is proportional to the price index. Investment demand significantly influences the interest rate.

Now we shall start with the nullcline point analysis.

Nullclines

In this section, we aim to determine from the above system of equations, the nullcline points. This will be achieved for x, y and z directions. It is worth noting that, nullcline points are different from equilibrium points since geometrically, they are points where vectors are either straight up or straight down. By definition of the $x_i - nullcline$ is defined as

$$ f_{x_i}(x_{1,\dots,}x_i,\dots,x_n) = 0. \tag{12}$$

In our case $x - nullcline$ points will be determined by imposing

$$ gz + (y-a)x = 0, \tag{13}$$

$y - nullcline$ points will be obtained with

$$ -by^3 - sx^2 + r = 0, \tag{14}$$

and $z - nullcline$ points will be obtained with

$$ -cz - \beta x - py = 0. \tag{15}$$

Indeed these points are different to equilibrium point. For the $x - nullcline$ points we have the following set

$$\begin{aligned} &(0,y,0),\ \forall y \in R_+, \\ &(x,a,0),\ \forall x \in R_+. \end{aligned} \tag{16}$$

For the $y - nullcline$ points, we have that if $x = y$, we shall have $-by^3 - sx^2 + r = 0$ the solution of the above equation will provide the set of nullcline (x,y,z), $x \neq 0$. For the $z - nullcline$ points, we have first $(0,0,0)$, $\left(x, x, \frac{-(p+\beta)}{c}x\right)$, if $x \neq 0$, many similar can be obtained.

Downward spikes

In this section, since we have (x,y,z), we shall have the Poincare section. Let us consider $L : y = mx + c$ for 3 cases. The first case we consider (x,y) which is the projection on the plane (x,y). The second will be (x,z) and the last is (y,z). We note that we have $y = mx + c$ is a Poincare section chosen to intersect the trajectories transversely, which implies that, no trajectory will be tangential to L. We recall that a Poincare mapping is defined as

$$ \Omega_{n+1} = \Gamma\left(\Omega_n\right), \tag{17}$$

where Ω_n, $n = 0,1,2,\dots,n$ consists of the sequence of coordinates located in L. Here the line and a trajectory of the system will intersect. Without loss of generality we provide for (x,y). We shall first solve $\left(x', y'\right)$ numerically to obtain $\left(x\left(t_k\right), y\left(t_k\right)\right)$, then we shall construct

$$ y\left(t_k\right) = mx\left(t_k\right) + c, \tag{18}$$

the procedure used here is from [14]. Then the following formula provides the downward spikes that will occur wherever

$$ y\left(t_k\right) - mx\left(t_k\right) - c = 0. \tag{19}$$

But to obtain these coordinates we shall use the predictor–corrector midpoint approach or the Heun's method.

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y),\ x(t_0) = x_0, \\ \frac{dy}{dt} &= f_1(t,x,y),\ y(t_0) = y_0, \end{aligned} \tag{20}$$

$$\begin{aligned} x_k &= x_{k-1} + \frac{h}{2}\left(f\left(x_k, y_k\right) + f\left(x_k + h, y_k + hf\left(x_k, y_k\right)\right)\right), \\ y_k &= y_{k-1} + \frac{h}{2}\left(f_1\left(x_k, y_k\right) + f_1\left(x_k + h, y_k + hf_1\left(x_k, y_k\right)\right)\right). \end{aligned} \tag{21}$$

Finally the downward spike is given as

$$ D(t_k) = \log\left|y\left(t_k\right) - mx\left(t_k\right) - c\right|. \tag{22}$$

Some numerical simulations are presented below.

Dynamical behavior of the model

In [8] the following equilibrium points were obtained for given model where

$$c = f = \beta = 1, s = 0.1, p = 0.05, g = 1.2, \tag{23}$$
$$E_1 = (0.04949849664, -7.070201517, 0.3040115792),$$
$$E_2 = (0.07616084207, 7.069016737, -0.4296116789),$$
$$E_3 = (3.087391472, 1.529728564, -3.163877901),$$
$$E_4 = (-3.093050811, 1.471456216, 3.019478000).$$

It was found in [8] that all the above equilibrium points are unable that is to say the eigenvalues have nonzero real part, therefore the nonlinear flow is conjugate to the flow of the linearized system in the neighborhood of E_1, E_2, E_3 and E_4. However different behaviors of the dynamic of this model can be obtained particular values of g, r. For example for $r = 0$, we have the following model

$$\frac{dx}{dt} = gz + (y-a)x, \tag{24}$$
$$\frac{dy}{dt} = -by^3 - sx^2,$$
$$\frac{dz}{dt} = -cz - \beta x - py.$$

Here $(0,0,0)$ is an equilibrium point. We shall also note that

$$f_1(x,y,z,t) = gz + (y-a)x, \tag{25}$$
$$f_2(x,y,z,t) = -by^3 - sx^2,$$
$$f_3(x,y,z,t) = -cz - \beta x - py.$$

$$f_1(0,0,0) = f_2(0,0,0) = f_3(0,0,0) = 0 \tag{26}$$

is a fixed point.

If $g = 0$, then we shall have

$$\frac{dx}{dt} = (y-a)x, \tag{27}$$
$$\frac{dy}{dt} = -by^3 - sx^2 + r,$$
$$\frac{dz}{dt} = -cz - \beta x - py.$$

We obtain the following equilibrium points if $r \neq 0$

$$y = a, -ba^3 - sx^{*2} + r = 0, -cz^* - \beta x^* - py^* = 0, \tag{28}$$
$$x^{*2} = \frac{r - ba^3}{s},\ z^* = -\frac{\beta x^* + pa}{c}.$$

If $r = ba^3$, then

$$x^* = 0,\ y^* = a,\ z^* = -\frac{pa}{c}. \tag{29}$$

If $\frac{r-ba^3}{s} > 0$, then

$$x^* = \mp\sqrt{\frac{r - ba^3}{s}},\ y^* = a,\ z^*_{1,2} = \frac{x^*_{1,2} + pa}{c}. \tag{30}$$

The Jacobian associate to this system is given by

$$J = \begin{bmatrix} -ay & x & g \\ -2sx & -3by^2 & 0 \\ -\beta & -p & -c \end{bmatrix}. \tag{31}$$

If $r = 0$, then $J_{r=0}$ can be obtained as

$$J^{E^*}_{r=0} = \begin{bmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ -\beta & -p & -c \end{bmatrix}. \tag{32}$$

$$\det\left[J^{E^*}_{r=0} - \lambda I\right] = \begin{bmatrix} -\lambda_1 & 0 & g \\ 0 & -\lambda_2 & 0 \\ -\beta & -p & -c-\lambda_3 \end{bmatrix}. \tag{33}$$

$\lambda_2 = 0$ and $\lambda_1 = \frac{-g\beta}{c+\lambda_3}$. Since $g, \beta, c > 0$ then $\lambda_1 + \frac{a}{b+\lambda_3} = 0$ provide a set of eigenvalues verifying the above equations. If $r \neq 0$ and $g = 0$, we have

$$J_{g=0}\left(x^*, y^*, z^*\right) = \begin{bmatrix} -ay^* & x^* & 0 \\ -2sx^* & -3by^{*2} & 0 \\ -\beta & -p & -c \end{bmatrix}, \tag{34}$$

$$J_{g=0}\left(0, a, \frac{-pa}{c}\right) = \begin{bmatrix} -a^2 & 0 & 0 \\ 0 & -3ba^2 & 0 \\ -\beta & -p & -c \end{bmatrix},$$

$$\det\left[J_{g=0}\left(0, a, \frac{-pa}{c}\right) - \lambda I\right] = 0,$$
$$\left(-a^2 - \lambda_1\right)\left(-3ba^2 - \lambda_2\right)\left(-c - \lambda_3\right) = 0.$$

In this case,

$$\lambda_1 = -a^2, \lambda_2 = -3ba^2, \lambda_3 = -c, \tag{35}$$

which are all negative which leads to stability. However at least we recall that, we are dealing with financial model therefore some restrictions could be set to avoid multiple scenarios.

Global stability of the equilibrium point of financial model

We have obtained above equilibrium points for the considered mathematical model. In this section, we aim at providing a theoretical behavior of these equilibrium points using existing theoretical methods. In this section, we will search for global stability of given model. Let us consider model again:

$$\frac{dx}{dt} = gz + (y-a)x, \tag{36}$$
$$\frac{dy}{dt} = -by^3 - sx^2 + r,$$
$$\frac{dz}{dt} = -cz - \beta x - py.$$

Theorem 3. *The system is globally asymptotically stable if the equilibrium point $E^*(x^*, y^*, z^*)$ satisfy the following conditions.*

$$T_1 - T_2 > 0 \text{ then } \frac{dL(t)}{dt} > 0, \tag{37}$$
$$T_1 - T_2 = 0 \text{ then } \frac{dL(t)}{dt} = 0,$$
$$T_1 - T_2 < 0 \text{ then } \frac{dL(t)}{dt} < 0.$$

Here

$$T_1 = gz + yx + x^*a + r + y^*by^2 + \frac{y^*}{y}sx^2 + z^*c + \frac{z^*}{z}\beta x + \frac{z^*}{z}py \tag{38}$$

and

$$T_2 = ax + \frac{x^*}{x}gz + x^*y + by^3 + sx^2 + \frac{y^*}{y}r + cz + \beta x + py. \tag{39}$$

Proof. We prove this using the idea of Lyapunov function. We start by defining the Lyapunov function associated the system as below:

$$L(E^*(x^*, y^*, z^*)) = \left(x - x^* + x^*\log\frac{x^*}{x}\right) + \left(y - y^* + y^*\log\frac{y^*}{y}\right) + \left(z - z^* + z^*\log\frac{z^*}{z}\right). \tag{40}$$

By the derivative of Lyapunov function with respect to t, we get

$$\frac{dL(t)}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt}. \tag{41}$$

Now we put values in above equation for derivatives

$$\frac{dL(t)}{dt} = \left(\frac{x-x^*}{x}\right)(gz+(y-a)x) + \left(\frac{y-y^*}{y}\right)\left(-by^3-sx^2+r\right) + \left(\frac{z-z^*}{z}\right)(-cz-\beta x-py). \tag{42}$$

Then we write

$$\frac{dL(t)}{dt} = gz+yx-ax-\frac{x^*}{x}gz-\frac{x^*}{x}yx+\frac{x^*}{x}ax - by^3-sx^2+r+\frac{y^*}{y}by^3+\frac{y^*}{y}sx^2-\frac{y^*}{y}r - cz-\beta x-py+\frac{z^*}{z}cz+\frac{z^*}{z}\beta x+\frac{z^*}{z}py. \tag{43}$$

Let us write above also

$$\frac{dL(t)}{dt} = T_1 - T_2, \tag{44}$$

here

$$T_1 = gz+yx+x^*a+r+y^*by^2+\frac{y^*}{y}sx^2 + z^*c+\frac{z^*}{z}\beta x+\frac{z^*}{z}py \tag{45}$$

and

$$T_2 = ax+\frac{x^*}{x}gz+x^*y+by^3+sx^2 + \frac{y^*}{y}r+cz+\beta x+py. \tag{46}$$

Therefore if

$$T_1-T_2>0 \text{ then } \frac{dL(t)}{dt}>0, \quad T_1-T_2=0 \text{ then } \frac{dL(t)}{dt}=0, \quad T_1-T_2<0 \text{ then } \frac{dL(t)}{dt}<0. \tag{47}$$

Existence and uniqueness

The mathematical model under consideration, is highly nonlinear, thus, with the current available knowledge, there is no suitable mathematical method that can help obtain analytical solutions to this system as in the case of many nonlinear equations. Researchers in this case commonly use numerical scheme to provide numerical solutions to these system, which is also the case of our system. Nevertheless, before providing this numerical solutions, we have to provide conditions under which these system admit a system of unique solutions. This will be achieved by evaluation the continuity of partial derivatives of respective functions forming the model. We verify in this section the criteria under which the above system admits a unique system of solutions. Let us take partial derivatives of equations.

$$\begin{aligned} &\tfrac{\partial f_1(x,y,z,t)}{\partial x}=(y-a), && \tfrac{\partial f_1(x,y,z,t)}{\partial y}=x, && \tfrac{\partial f_1(x,y,z,t)}{\partial z}=g,\\ &\tfrac{\partial^2 f_1(x,y,z,t)}{\partial x^2}=0, && \tfrac{\partial^2 f_1(x,y,z,t)}{\partial y^2}=0, && \tfrac{\partial^2 f_1(x,y,z,t)}{\partial z^2}=0,\\ &\tfrac{\partial^2 f_1(x,y,z,t)}{\partial x\partial y}=1, && \tfrac{\partial^2 f_1(x,y,z,t)}{\partial x\partial z}=0, && \tfrac{\partial^2 f_1(x,y,z,t)}{\partial y\partial z}=0, \end{aligned} \tag{48}$$

We verify the same with f_2 and f_3,

$$\begin{aligned} &\tfrac{\partial f_2(x,y,z,t)}{\partial x}=-2sx, && \tfrac{\partial f_2(x,y,z,t)}{\partial y}=-3by^2, && \tfrac{\partial f_2(x,y,z,t)}{\partial z}=0,\\ &\tfrac{\partial^2 f_2(x,y,z,t)}{\partial x^2}=-2s, && \tfrac{\partial^2 f_2(x,y,z,t)}{\partial y^2}=-6by, && \tfrac{\partial^2 f_2(x,y,z,t)}{\partial z^2}=0,\\ &\tfrac{\partial^2 f_2(x,y,z,t)}{\partial x\partial y}=0, && \tfrac{\partial^2 f_2(x,y,z,t)}{\partial x\partial z}=0, && \tfrac{\partial^2 f_2(x,y,z,t)}{\partial z\partial y}=0, \end{aligned} \tag{49}$$

$$\begin{aligned} &\tfrac{\partial f_3(x,y,z,t)}{\partial x}=-\beta, && \tfrac{\partial f_3(x,y,z,t)}{\partial y}=-p, && \tfrac{\partial f_3(x,y,z,t)}{\partial z}=-c,\\ &\tfrac{\partial^2 f_3(x,y,z,t)}{\partial x^2}=0, && \tfrac{\partial^2 f_3(x,y,z,t)}{\partial y^2}=0, && \tfrac{\partial^2 f_3(x,y,z,t)}{\partial z^2}=0,\\ &\tfrac{\partial^2 f_3(x,y,z,t)}{\partial x\partial y}=0, && \tfrac{\partial^2 f_3(x,y,z,t)}{\partial x\partial z}=0, && \tfrac{\partial^2 f_3(x,y,z,t)}{\partial z\partial y}=0, \end{aligned} \tag{50}$$

We have verified that

$$\forall(x,y,z,t)\in R_0=\left\{\left|t-t_0\right|<a,\ \left|x-x_0\right|<b_1,\ \left|y-y_0\right|<b_2,\ \left|z-z_0\right|<b_3\right\}, \tag{51}$$

the partial derivative of f_1, f_2 and f_3 are continuously differentiable. Thus under an additional condition that $\forall(x,y,z,t)\in R_0$, f_1, f_2 and f_3 are bounded, then we can construct the following Tonelli sequences [15]

$$\begin{aligned} &x_n=x_0, \quad if\ t\in\left[0,\frac{1}{n}\right]\\ &y_n=y_0,\\ &z_n=z_0,\\ &x_n=x_0+\int_0^{t-\frac{1}{n}} f_1\left(x_n,y_n,z_n,\tau\right)d\tau, \quad if\ \frac{1}{n}\le t\le\frac{2}{n}\\ &y_n=y_0+\int_0^{t-\frac{1}{n}} f_2\left(x_n,y_n,z_n,\tau\right)d\tau,\\ &z_n=z_0+\int_0^{t-\frac{1}{n}} f_3\left(x_n,y_n,z_n,\tau\right)d\tau. \end{aligned} \tag{52}$$

Due to the boundness of the functions f_1, f_2 and f_3 are the continuity of these functions, the above sequence admits a subsequence $\left(x_{nl},y_{nl},z_{nl}\right)_{n\ge0}$ of $\left(x_n,y_n,z_n\right)_{n\ge0}$ that converges, since $\left(x_n,y_n,z_n\right)_{n\ge0}$ is equicontinuous and uniformly bounded. The uniqueness is obtained due to the fact that f_1, f_2 and f_3 are Lipschitz under $\left[t_0,t_0+\lambda\right]$ where $\lambda=\min\left\{a,\frac{b}{M}\right\}$ where $\overline{M}=\max\left\{M_{f_1},M_{f_2},M_{f_3}\right\}$,

$$M_{f_i}=\max_{t\in[t,t_0+a]}\left|f_i(x,y,z,t)\right|. \tag{53}$$

Alternatively,

$$\overline{x}=\max\left\{x_1,x_2,x_3\right\}, \quad x_1=\max_{t\in[t,t_0+a]}|x(t)|,\ x_2=\max_{t\in[t,t_0+a]}|y(t)|,\ x_3=\max_{t\in[t,t_0+a]}|z(t)|. \tag{54}$$

$$\begin{aligned} \left|f_1(x,y,z,t)\right| &\le g\overline{x}+\overline{x}|x|,\\ &\le g\overline{x}\left(1+\frac{1}{g}|x|\right),\\ &\le k_1\left(1+\frac{1}{g}|x|\right),\\ &\le k_1(1+|x|), \end{aligned} \tag{55}$$

if

$$\frac{1}{g}<1. \tag{56}$$

$$\begin{aligned} \left|f_2(x,y,z,t)\right| &\le b\left|y^3\right|-s\overline{x}^2+r,\\ &\le b\overline{x}^2|y|+s\overline{x}^2+r,\\ &\le\left(s\overline{x}^2+r\right)\left(1+\frac{b\overline{x}^2|y|}{s\overline{x}^2+r}\right),\\ &\le\left(s\overline{x}^2+r\right)(1+|y|), \end{aligned} \tag{57}$$

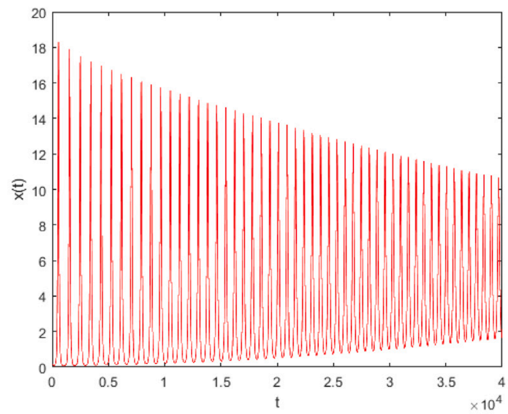
if

$$\frac{b\overline{x}^2}{s\overline{x}^2+r}<1. \tag{58}$$

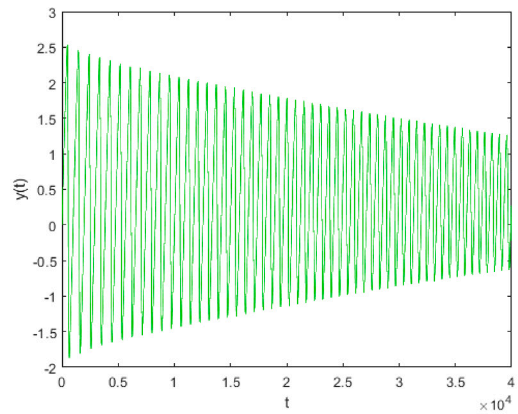
$$\begin{aligned} \left|f_3(x,y,z,t)\right| &\le c|z|+\beta\overline{x}+p\overline{x}\\ &\le\overline{x}(p+\beta)\left(1+\frac{c}{\overline{x}(p+\beta)}|z|\right)\\ &\le\overline{x}(p+\beta)(1+|z|), \end{aligned} \tag{59}$$

if

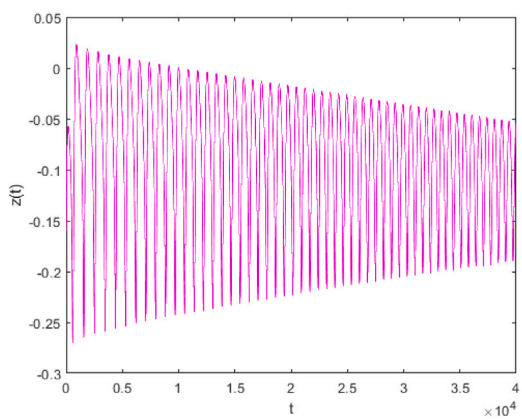
$$\frac{c}{\overline{x}(p+\beta)}<1. \tag{60}$$



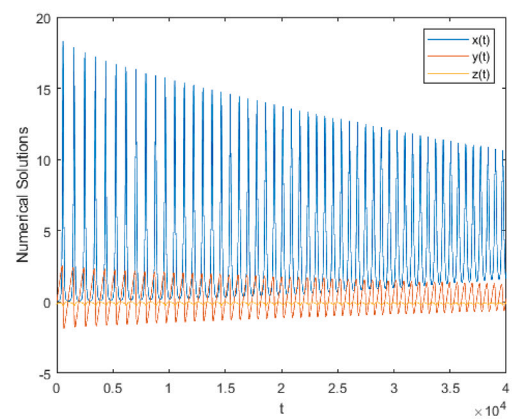
(a) Numerical simulation for $x(t)$.



(b) Numerical simulation for $y(t)$.



(c) Numerical simulation for $z(t)$.



(d) Numerical simulations for $x(t)$, $y(t)$ and $z(t)$.

Fig. 1. Numerical simulation results of system for classical case.

Taking

$$max\left\{\frac{1}{g},\frac{b\overline{x}^{2}}{s\overline{x}^{2}+r},\frac{c}{\overline{x}\left(p+\beta\right)}\right\}=\Omega. \tag{61}$$

If $\Omega<1$ then the system verifying the Caratheodory principle that leads to existence of the solution [16]. The existence and uniqueness allow us to derive a numerical solution since the system is nonlinear.

The study presented above is based on differential operator that uses only the rate of change therefore only a class of some financial behaviors can be captured. However, financial behavior portrait different dynamics that in our opinion cannot be replicated using a differential operator based on the rate of change. In the next section, we shall reformulate the problem by replacing the classical derivative with nonlocal operator.

Model with Caputo–Fabrizio derivative

Processes with fading memory could be observed in financial dynamic for example inflation. To capture such dynamic, we replace by the Caputo–Fabrizio fractional derivative to obtain

$$\begin{aligned}
{}_{0}^{CF}D_{t}^{\alpha}x(t) &= f_{1}\left(x,y,z,t\right) \qquad if\ t>0,\\
{}_{0}^{CF}D_{t}^{\alpha}y(t) &= f_{2}\left(x,y,z,t\right) \qquad if\ t>0,\\
{}_{0}^{CF}D_{t}^{\alpha}z(t) &= f_{3}\left(x,y,z,t\right) \qquad if\ t>0,\\
x\left(0\right) &= x_{0},\\
y\left(0\right) &= y_{0},\\
z\left(0\right) &= z_{0}.
\end{aligned} \tag{62}$$

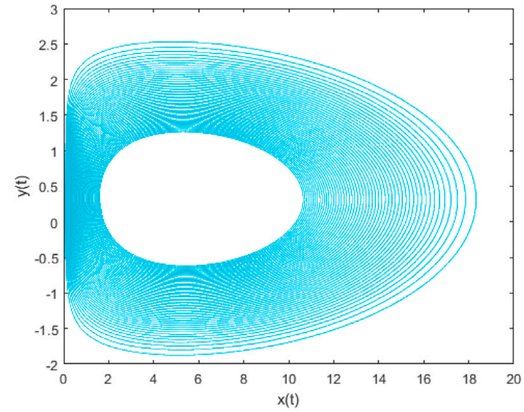
f_1, f_2 and f_3 are the same with the same condition like in the classical case. Here to achieve uniqueness, we need

$$\lambda=\min\left\{a,\frac{b}{M}\right\}. \tag{63}$$

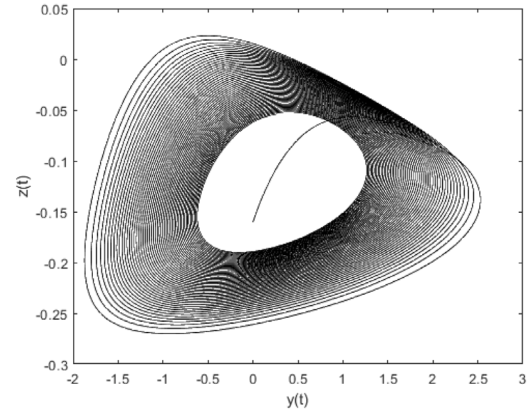
The following Picard iteration

$$\begin{aligned}
x_{n}(t) &= x(0)+(1-\alpha)\,f_{1}\left(t,x_{n-1},y_{n-1},z_{n-1}\right)\\
&\quad+\alpha\int_{0}^{t}f_{1}\left(\tau,x_{n-1}\left(\tau\right),y_{n-1}\left(\tau\right),z_{n-1}\left(\tau\right)\right)d\tau,\\
y_{n}(t) &= y(0)+(1-\alpha)\,f_{2}\left(t,x_{n-1},y_{n-1},z_{n-1}\right)\\
&\quad+\alpha\int_{0}^{t}f_{2}\left(\tau,x_{n-1}\left(\tau\right),y_{n-1}\left(\tau\right),z_{n-1}\left(\tau\right)\right)d\tau,\\
z_{n}(t) &= z(0)+(1-\alpha)\,f_{3}\left(t,x_{n-1},y_{n-1},z_{n-1}\right)\\
&\quad+\alpha\int_{0}^{t}f_{3}\left(\tau,x_{n-1}\left(\tau\right),y_{n-1}\left(\tau\right),z_{n-1}\left(\tau\right)\right)d\tau,
\end{aligned} \tag{64}$$

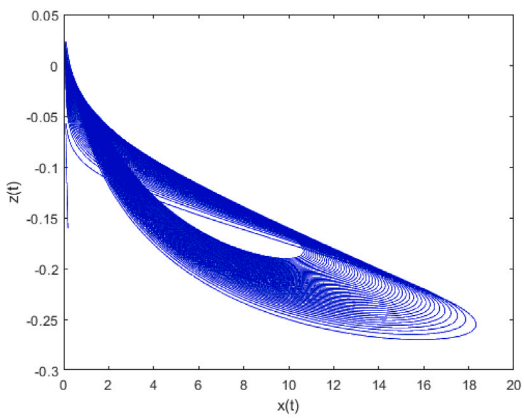
are equicontinuous and uniformly bounded. Therefore they admit a system of subsequences $\left(x_{nl}\right),\left(y_{nl}\right)$ and $\left(z_{nl}\right)$ that convergence to the solution of the system of the equations. The uniqueness is achieved by either using the fact that the function f_1, f_2 and f_3 satisfy the Caratheodory principles or the Lipschitz conditions. This therefore leads us to a numerical scheme. We shall adopt the previous method.



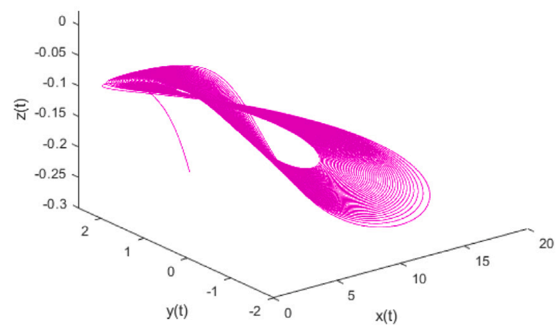
(e) Numerical simulations for $x-y$ phase.



(f) Numerical simulations for $y-z$ phase.



(g) Numerical simulations for $x-z$ phase.



(h) Numerical simulations for $x-y-z$ phase.

Fig. 2. Numerical simulation results of system for classical case.

$$\begin{aligned}
x_{n+1} &= x_n + (1-\alpha)\left[f_1\left(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}\right) - f_1\left(t_n, x_n, y_n, z_n\right)\right] \\
&\quad + 2h\alpha f_1\left(t_n, x_n, y_n, z_n\right), \\
y_{n+1} &= y_n + (1-\alpha)\left[f_2\left(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}\right) - f_2\left(t_n, x_n, y_n, z_n\right)\right] \\
&\quad + 2h\alpha f_2\left(t_n, x_n, y_n, z_n\right), \\
z_{n+1} &= z_n + (1-\alpha)\left[f_3\left(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}\right) - f_3\left(t_n, x_n, y_n, z_n\right)\right] \\
&\quad + 2h\alpha f_3\left(t_n, x_n, y_n, z_n\right).
\end{aligned} \tag{65}$$

We can notice that the above is implicit, to make it explicit we replace the x_{n+1}, y_{n+1} and z_{n+1} of the right hand side by

$$\begin{aligned}
x_{n+1} &= x_0 + (1-\alpha) f_1\left(t_n, x_n, y_n, z_n\right) + h\alpha \sum_{j=0}^{n} f_1\left(t_j, x_j, y_j, z_j\right), \\
y_{n+1} &= y_0 + (1-\alpha) f_2\left(t_n, x_n, y_n, z_n\right) + h\alpha \sum_{j=0}^{n} f_2\left(t_j, x_j, y_j, z_j\right), \\
z_{n+1} &= z_0 + (1-\alpha) f_3\left(t_n, x_n, y_n, z_n\right) + h\alpha \sum_{j=0}^{n} f_3\left(t_j, x_j, y_j, z_j\right).
\end{aligned} \tag{66}$$

Model with power law process

In order to include into the mathematical formulation of the new financial model the effect of power-law, we replace the classical derivative with the Caputo derivative to obtain

$$\begin{aligned}
{}_0^C D_t^\alpha x(t) &= f_1(x, y, z, t) \qquad if\ t > 0, \\
{}_0^C D_t^\alpha y(t) &= f_2(x, y, z, t) \qquad if\ t > 0, \\
{}_0^C D_t^\alpha z(t) &= f_3(x, y, z, t) \qquad if\ t > 0, \\
x(0) &= x_0, \\
y(0) &= y_0, \\
z(0) &= z_0.
\end{aligned} \tag{67}$$

$$\begin{aligned}
\frac{1}{\Gamma(1-\alpha)} \int_0^t x'(\tau)(t-\tau)^{-\alpha} d\tau &= f_1(x, y, z, t), \\
\frac{1}{\Gamma(1-\alpha)} \int_0^t y'(\tau)(t-\tau)^{-\alpha} d\tau &= f_2(x, y, z, t), \\
\frac{1}{\Gamma(1-\alpha)} \int_0^t z'(\tau)(t-\tau)^{-\alpha} d\tau &= f_3(x, y, z, t).
\end{aligned} \tag{68}$$

We transform the above into

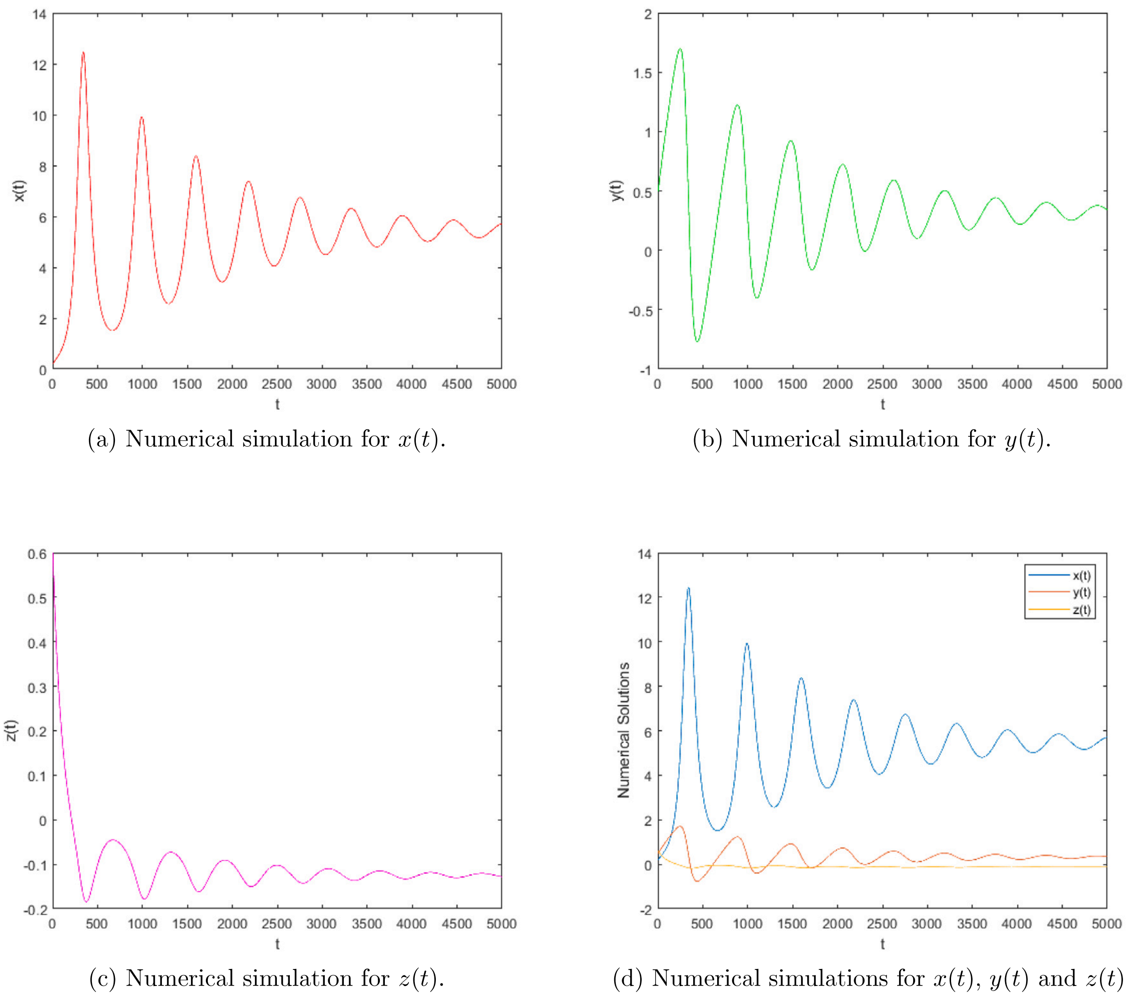
$$\begin{aligned}
x(t) &= x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f_1(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau, \\
y(t) &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f_2(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau, \\
z(t) &= z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f_3(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau.
\end{aligned} \tag{69}$$

For existence and uniqueness, we need only

$$\lambda = \min\left\{ a, \left(\frac{b\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}} \right\}. \tag{70}$$

Then the following predictor–corrector iterative formula can be utilized

$$x_{n+1}(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f_1\left(x_{n+1}, y_{n+1}, z_{n+1}, \tau\right)(t-\tau)^{\alpha-1} d\tau, \tag{71}$$



(a) Numerical simulation for $x(t)$.

(b) Numerical simulation for $y(t)$.

(c) Numerical simulation for $z(t)$.

(d) Numerical simulations for $x(t)$, $y(t)$ and $z(t)$.

Fig. 3. Numerical simulation results of system for Caputo case.

$$y_{n+1}(t)=y(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_2\left(x_{n+1},y_{n+1},z_{n+1},\tau\right)(t-\tau)^{\alpha-1}d\tau,$$
$$z_{n+1}(t)=z(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_3\left(x_{n+1},y_{n+1},z_{n+1},\tau\right)(t-\tau)^{\alpha-1}d\tau.$$

The system is implicit therefore

$$\begin{aligned}x_{n+1}(t)&=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_1\left(\overline{x}_{n+1},\overline{y}_{n+1},\overline{z}_{n+1},\tau\right)(t-\tau)^{\alpha-1}d\tau,\\ y_{n+1}(t)&=y(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_2\left(\overline{x}_{n+1},\overline{y}_{n+1},\overline{z}_{n+1},\tau\right)(t-\tau)^{\alpha-1}d\tau,\\ z_{n+1}(t)&=z(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_3\left(\overline{x}_{n+1},\overline{y}_{n+1},\overline{z}_{n+1},\tau\right)(t-\tau)^{\alpha-1}d\tau,\end{aligned}\tag{72}$$

where

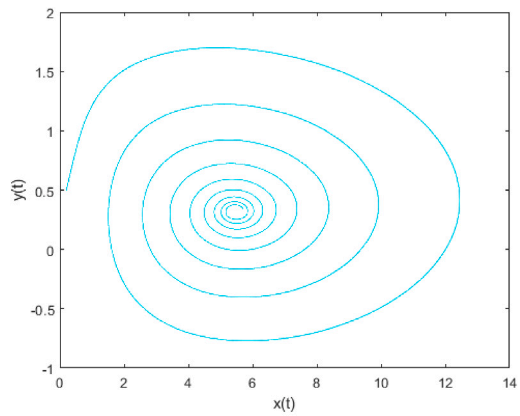
$$\begin{aligned}\overline{x}_{n+1}&=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_1\left(x_n,y_n,z_n,\tau\right)(t-\tau)^{\alpha-1}d\tau,\\ \overline{y}_{n+1}&=y(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_2\left(x_n,y_n,z_n,\tau\right)(t-\tau)^{\alpha-1}d\tau,\\ \overline{z}_{n+1}&=z(0)+\frac{1}{\Gamma(\alpha)}\int_0^t f_3\left(x_n,y_n,z_n,\tau\right)(t-\tau)^{\alpha-1}d\tau.\end{aligned}\tag{73}$$

This system is very accurate as it has been introduced in [17] with the aim to improve the Picard iteration. Since this is subsequence of the Picard iteration under the condition that $\left(f_i\right)_{i=1,2,3}$ are bounded and differentiable continuous it followed that if converges. The uniqueness is also satisfied via the Gronwall inequality since the functions f_1, f_2 and f_3 satisfy the Lipschitz condition. With the existence and uniqueness obtained we can now proceed with numerical solution via the Midpoint principle as follow

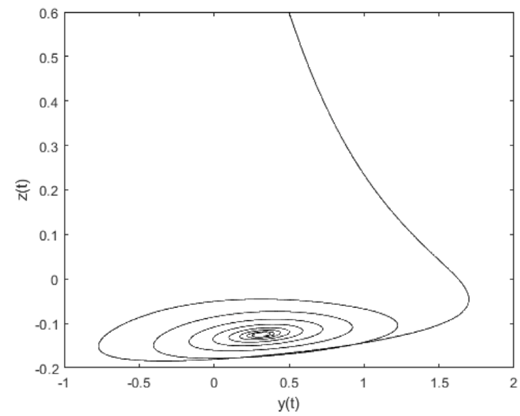
$$\begin{aligned}x_{n+1}(t)&=x(0)+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{n-1}f_1\left(t_j+\frac{h}{2},\frac{x_j+x_{j+1}}{2},\frac{y_j+y_{j+1}}{2},\frac{z_j+z_{j+1}}{2}\right)\\ &\quad\times\left\{(n-j+1)^\alpha-(n-j)^\alpha\right\}\\ &\quad+\frac{h^\alpha}{\Gamma(\alpha+1)}f_1\left(t_n+\frac{h}{2},\frac{x_n+\overline{x}_{n+1}}{2},\frac{y_n+\overline{y}_{n+1}}{2},\frac{z_n+\overline{z}_{n+1}}{2}\right),\\ y_{n+1}(t)&=y(0)+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{n-1}f_2\left(t_j+\frac{h}{2},\frac{x_j+x_{j+1}}{2},\frac{y_j+y_{j+1}}{2},\frac{z_j+z_{j+1}}{2}\right)\\ &\quad\times\left\{(n-j+1)^\alpha-(n-j)^\alpha\right\}\\ &\quad+\frac{h^\alpha}{\Gamma(\alpha+1)}f_2\left(t_n+\frac{h}{2},\frac{x_n+\overline{x}_{n+1}}{2},\frac{y_n+\overline{y}_{n+1}}{2},\frac{z_n+\overline{z}_{n+1}}{2}\right),\\ z_{n+1}(t)&=z(0)+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{n-1}f_3\left(t_j+\frac{h}{2},\frac{x_j+x_{j+1}}{2},\frac{y_j+y_{j+1}}{2},\frac{z_j+z_{j+1}}{2}\right)\\ &\quad\times\left\{(n-j+1)^\alpha-(n-j)^\alpha\right\}\\ &\quad+\frac{h^\alpha}{\Gamma(\alpha+1)}f_3\left(t_n+\frac{h}{2},\frac{x_n+\overline{x}_{n+1}}{2},\frac{y_n+\overline{y}_{n+1}}{2},\frac{z_n+\overline{z}_{n+1}}{2}\right),\end{aligned}\tag{74}$$

where

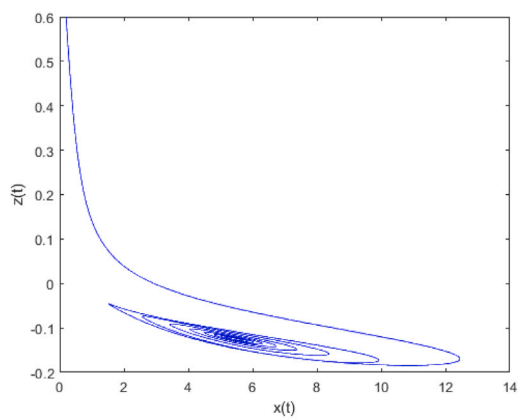
$$\overline{x}_{n+1}=x(0)+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{n}f_1\left(t_j,x_j,y_j,z_j\right)\left\{(n-j+1)^\alpha-(n-j)^\alpha\right\},\tag{75}$$
$$\overline{y}_{n+1}=y(0)+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{n}f_2\left(t_j,x_j,y_j,z_j\right)\left\{(n-j+1)^\alpha-(n-j)^\alpha\right\},$$



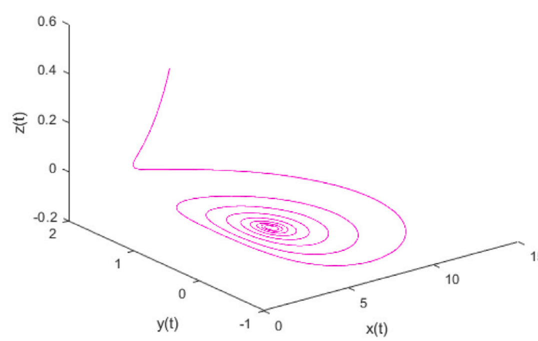
(e) Numerical simulations for $x-y$ phase.



(f) Numerical simulations for $y-z$ phase.



(g) Numerical simulations for $x-z$ phase.



(h) Numerical simulations for $x-y-z$ phase.

Fig. 4. Numerical simulation results of system for Caputo case.

$$\overline{z}_{n+1} = z(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_3\left(t_j, x_j, y_j, z_j\right)\left\{(n-j+1)^\alpha - (n-j)^\alpha\right\}.$$

Numerical simulations are presented below.

Numerical solution

In this section, we shall use a well-known accurate numerical scheme to solve the system. We shall adopt the Nystrom scheme [18]. We shall first show the general. Let $y' = f(t, y(t))$ be a general Cauchy problem with initial condition $y(t_0) = y_0$, f is continuous

$$\forall (t, y) \in R_0 = \left\{\left|t-t_0\right| < a,\ \left|y-y_0\right| < b\right\}. \tag{76}$$

The Nystrom scheme says

$$y_{n+1} = y_{n-1} + 2hf\left(t_n, y_n\right). \tag{77}$$

It is known that the method is of order $O\left(h^3\right)$. In our case, therefore

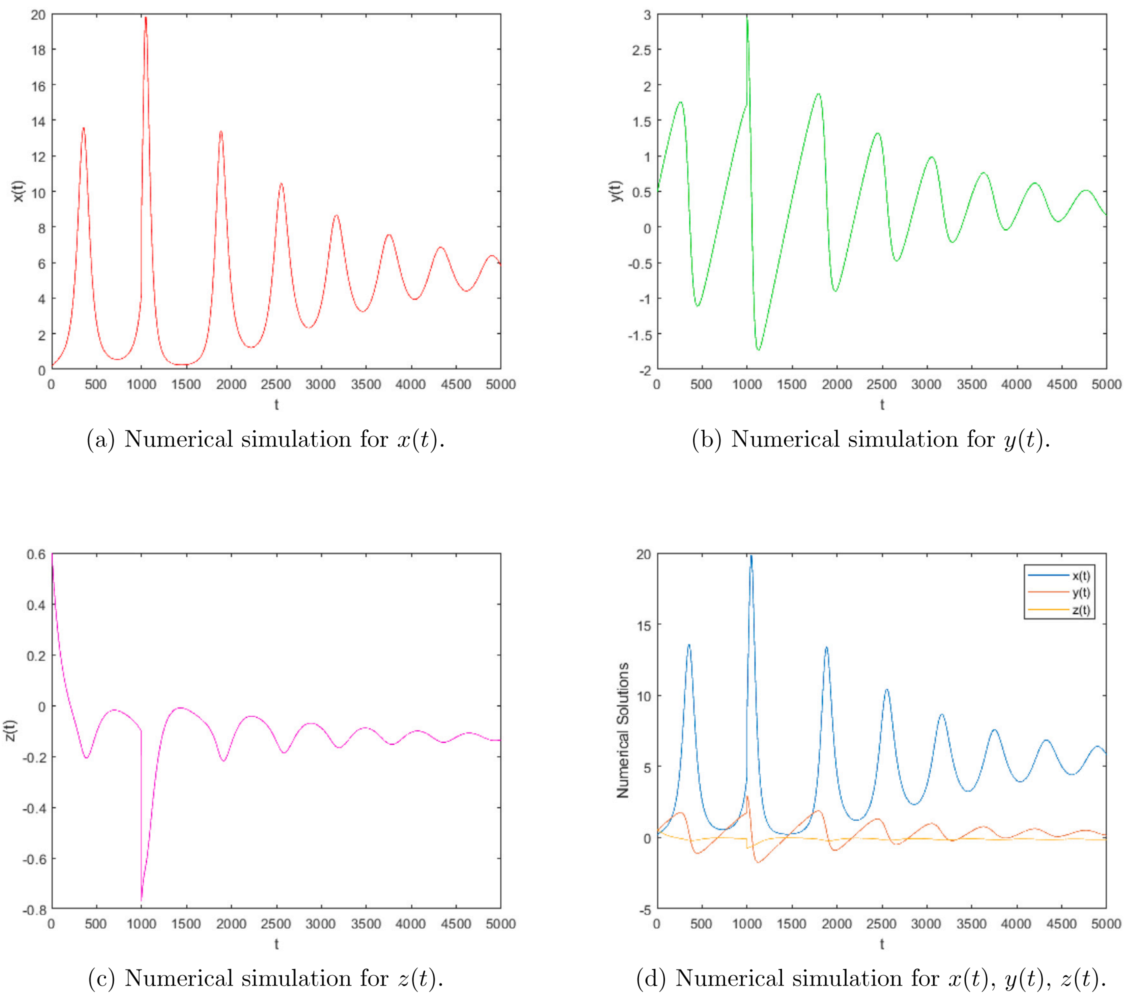
$$\begin{aligned} x_{n+1} &= x_{n-1} + 2hf_1\left(t_n, x_n, y_n, z_n\right), \\ y_{n+1} &= y_{n-1} + 2hf_2\left(t_n, x_n, y_n, z_n\right), \\ z_{n+1} &= z_{n-1} + 2hf_3\left(t_n, x_n, y_n, z_n\right). \end{aligned} \tag{78}$$

Numerical simulation will be depicted in following Figs. 1–2 given for different values of r and g for classical case.

The numerical solution of the model under investigation is therefore given as

$$\begin{aligned} x_{n+1}(t) &= x(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left\{ g\left(\frac{z_j + z_{j+1}}{2}\right) + \left(\left(\frac{y_j + y_{j+1}}{2}\right) - a\right) \times \left(\frac{x_j + x_{j+1}}{2}\right)\right\} \\ &\quad \times \left\{(n-j+1)^\alpha - (n-j)^\alpha\right\} \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+1)} \left\{ g\left(\frac{z_n + \overline{z}_{n+1}}{2}\right) + \left(\left(\frac{y_n + \overline{y}_{n+1}}{2}\right) - a\right) \times \left(\frac{x_n + \overline{x}_{n+1}}{2}\right)\right\}, \\ \overline{x}_{n+1} &= x(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \left\{ g z_j + \left(y_j - a\right) x_j \right\} \left\{(n-j+1)^\alpha - (n-j)^\alpha\right\} \end{aligned} \tag{79}$$

$$\begin{aligned} y_{n+1}(t) &= y(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left\{ -b\left(\frac{y_j + y_{j+1}}{2}\right)^3 - s\left(\frac{x_j + x_{j+1}}{2}\right)^2 + r \right\} \\ &\quad \times \left\{(n-j+1)^\alpha - (n-j)^\alpha\right\} \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+1)} \left\{ -b\left(\frac{y_n + \overline{y}_{n+1}}{2}\right)^3 - s\left(\frac{x_n + \overline{x}_{n+1}}{2}\right)^2 + r \right\}, \\ \overline{y}_{n+1} &= y(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \left\{ -by_j^3 - sx_j^2 + r \right\} \left\{(n-j+1)^\alpha - (n-j)^\alpha\right\} \end{aligned} \tag{80}$$



(a) Numerical simulation for $x(t)$.

(b) Numerical simulation for $y(t)$.

(c) Numerical simulation for $z(t)$.

(d) Numerical simulation for $x(t)$, $y(t)$, $z(t)$.

Fig. 5. Numerical simulation results of system for piecewise case.

$$
\begin{aligned}
z_{n+1}(t) = z(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \Bigg\{ -c\left(\frac{z_j+z_{j+1}}{2}\right) - \beta\left(\frac{x_j+x_{j+1}}{2}\right) \\
-p\left(\frac{y_j+y_{j+1}}{2}\right) \Bigg\} \\
\times \left\{ (n-j+1)^{\alpha} - (n-j)^{\alpha} \right\} \\
+ \frac{h^{\alpha}}{\Gamma(\alpha+1)} \Bigg\{ -c\left(\frac{z_n+\overline{z}_{n+1}}{2}\right) - \beta\left(\frac{x_n+\overline{x}_{n+1}}{2}\right) \\
-p\left(\frac{y_n+\overline{y}_{n+1}}{2}\right) \Bigg\}, \\
\overline{z}_{n+1} = z(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \left\{ -cz_j - \beta x_j - p y_j \right\} \left\{ (n-j+1)^{\alpha} - (n-j)^{\alpha} \right\}
\end{aligned}
\tag{81}
$$

Numerical simulation are presented below in Figs. 3 to 4 given for different values of r and g for Caputo case.

With using piecewise case idea [19,20], we have considered the following piecewise model

$$
\begin{cases}
x'(t) = f_1(x, y, z, t), & \text{if } t \in [0, t_1] \\
y'(t) = f_2(x, y, z, t), & \\
z'(t) = f_3(x, y, z, t) &
\end{cases},
\tag{82}
$$

$$
\begin{cases}
{}_{t_1}^{C}D_t^{\alpha} x(t) = f_1(x, y, z, t), & \text{if } t \in [t_1, T] \\
{}_{t_1}^{C}D_t^{\alpha} y(t) = f_2(x, y, z, t), & \\
{}_{t_1}^{C}D_t^{\alpha} z(t) = f_3(x, y, z, t) &
\end{cases}.
$$

The above is easily converted to

$$
\begin{cases}
x(t) = x(0) + \int_0^t f_1(x, y, z, \tau)\, d\tau, & \text{if } t \in [0, t_1] \\
y(t) = y(0) + \int_0^t f_2(x, y, z, \tau)\, d\tau, & \\
z(t) = z(0) + \int_0^t f_3(x, y, z, \tau)\, d\tau, &
\end{cases}
\tag{83}
$$

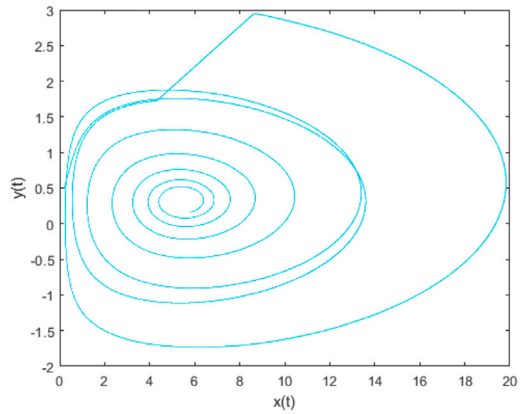
$$
\begin{cases}
x(t) = x(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t f_1(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau, & \text{if } t \in [t_1, T] \\
y(t) = y(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t f_2(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau, & \\
z(t) = z(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t f_3(x, y, z, \tau)(t-\tau)^{\alpha-1} d\tau, &
\end{cases}
\tag{84}
$$

Here we can use Nystrom scheme of the classical case and fractional Euler for the fractional case as follow,

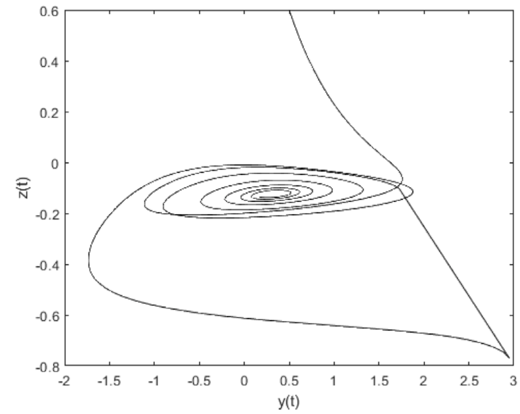
$$
\begin{cases}
x_{n+1} = x_{n-1} + 2hf_1(t_n, x_n, y_n, z_n), & \text{if } t_n \in [0, t_1] \\
y_{n+1} = y_{n-1} + 2hf_2(t_n, x_n, y_n, z_n), & \\
z_{n+1} = z_{n-1} + 2hf_3(t_n, x_n, y_n, z_n), &
\end{cases}.
\tag{85}
$$

$$
\begin{cases}
x_{n+1}(t) = x(t_1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_1(t_j, x_j, y_j, z_j) \left\{ (n-j+1)^{\alpha} - (n-j)^{\alpha} \right\}, & \text{if } t \in [t_1, T] \\
y_{n+1}(t) = y(t_1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_2(t_j, x_j, y_j, z_j) \left\{ (n-j+1)^{\alpha} - (n-j)^{\alpha} \right\}, & \\
z_{n+1}(t) = z(t_1) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_3(t_j, x_j, y_j, z_j) \left\{ (n-j+1)^{\alpha} - (n-j)^{\alpha} \right\}, &
\end{cases}.
$$

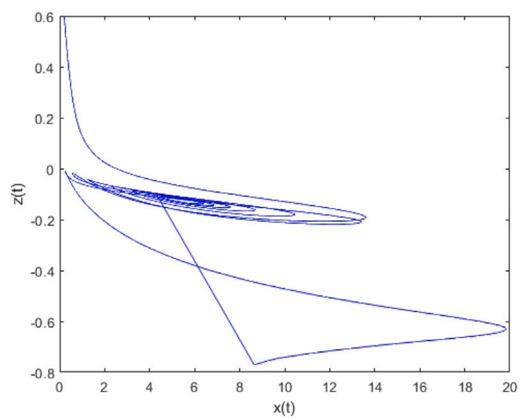
Numerical simulations are presented in Figs. 5–6 given for different values of r and g for piece-wise case.



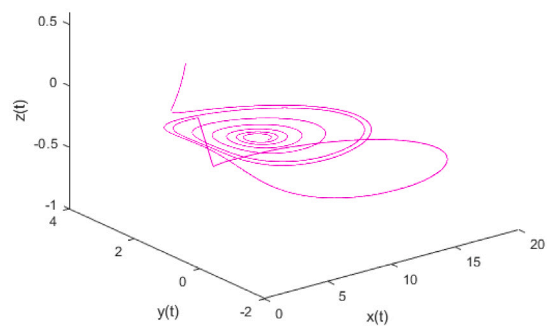
(e) Numerical simulations for $x - y$ phase.



(f) Numerical simulations for $y - z$ phase.



(g) Numerical simulations for $x - z$ phase.



(h) Numerical simulations for $x - y - z$ phase.

Fig. 6. Numerical simulation results of system for piecewise case.

Conclusion

In this work, we extend a new mathematical model with three nonlinear equations that was proposed to represent phenomena resembling those portrayed by financial processes. In this situation, the Poincare mapping was used to project the model into two dimensions. We also evaluated the stability of the equilibrium points and their associated Lyapunov functions and Poincare mapping. For many differential operators, we have offered conditions for the solutions, existence and uniqueness. We also used the Nystrom and midpoint concepts to solve the model.

CRediT authorship contribution statement

Ilknur Koca: Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article

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