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A numerical technique for solving functional integro-differential equations having variable bounds

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Abstract In this paper, a collocation method based on Taylor polynomials is presented to solve the functional delay integro-differential equations with variable bounds. Using this method, we transform the functional equations to a system of linear algebraic equations. Thus, the unknown coefficients of the approximate solution are determined by solving this system. An error analysis technique based on residual function is developed to improve the numerical solution. Some numerical examples are given to illustrate the accuracy and applicability of the method. Finally, the data are examined according to the residual error estimation. All numerical computations have been performed on the computer programs.

Keywords Functional integro-differential equations · Taylor polynomials · Collocation points · Approximate solutions · Residual error technique

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1 Introduction

Functional differential equations and integro-differential equations play important role for modeling problems in engineering, mechanics, physics, economics, and astronomy (Iserles and Liu 1994; Wu 2012; Kolmanovskii et al. 2013; Ali 2011; Brunner and Hu 2007). Since solving these problems analytically can be difficult, some numerical methods have been developed. Therefore, in recent years, there have been many studies on numerical methods of functional integro-differential equations (Brunner and van der Houwen 1986; Doha et al. 2014; Wang and Wang 2013, 2014; Bhrawy et al. 2013). Sahu and Ray (2015) have used Legendre spectral collocation method to solve Fredholm integro-differential-difference equation, Borhanifar and Sadri (2015) have presented an operational method based on Jacobi polynomials for numerical solution of generalized functional integro-differential equations, Wang and Li (2009) have studied on one-leg methods for nonlinear neutral delay integro-differential equations, Karakoç et al. (2013) have applied homotopy perturbation method to find approximate solution of Fredholm integro-differential-difference equations, and Rihan et al. (2009) have solved the Volterra delay integro-differential equations using the technique based on the mono-implicit Runge–Kutta method.

In addition to these methods, Volterra-type functional integral equations, pantograph-type integro-differential equations, and delay integro-differential difference equations have been solved using the Taylor collocation method (Gökmen et al. 2017), the Chelyshkov collocation method (Oguz and Sezer 2015), Dickson collocation method (Kürkçü et al. 2016), and Laguerre polynomial approach (Gürbüz et al. 2014) by Sezer and his colleagues.

In this article, we consider the functional integro-differential equations (FIDEs) with variable bounds and mixed delays represented by:

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)\, y^{(k)}(\alpha_{kj}x+\beta_{kj}) = f(x)+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\, y^{(r)}(\mu_{rs}t+\gamma_{rs})\mathrm{d}t, \quad m_1 \geq m_3, \tag{1}$$

under the mixed conditions:

$$\sum_{k=0}^{m_1-1}\left[a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)\right]=\eta_i, \quad i=0,1,\ldots,m_1-1, \tag{2}$$

where the known functions $P_{kj}(x)$, $K_{rs}(x,t)$, $f(x)$, $u_{rs}(x)$, $v_{rs}(x)$ are continuous on the interval $[a,b]$, $a \leq u_{rs}(x) < v_{rs}(x) \leq b$ and α_{kj}, β_{kj}, λ_{rs}, μ_{rs}, γ_{rs} are real constants.

The aim of this study is to obtain an approximate solution of problems (1) and (2) using in the truncated Taylor series form:

$$y(x) \cong y_N(x)=\sum_{n=0}^{N} y_n\, x^n, \quad y_n=\frac{y^{(n)}(0)}{n!}, \tag{3}$$

where y_n, $n=0,1,\ldots,N$ are the unknown coefficients and are determined and N is chosen any positive integers.

To explain our method, we have organized this paper as follows: Taylor matrix forms of each term of problems (1) and (2) have been given in Sect. 2. In Sect. 3, the Taylor collocation method has been described using these matrix forms based on the collocation points. In

Sect. 4, the error analysis technique based on the residual function has been developed for the present method. In Sect. 5, the numerical examples have been given to show the efficiency and applicability of the mentioned method. Finally, in Sect. 6, results have been obtained and the paper has been summarized.

2 Fundamental matrix relations

In this section, our aim is to convert Eq. (1) to a matrix equation. For this purpose, we construct the matrix forms of each term of Eq. (1). We first consider the approximate solution $y(x)$ and its derivative $y^{(k)}(x)$ defined by the truncated Taylor series (3). Then, we write (3) and its derivatives in the matrix form:

$$\mathbf{y}(x) = \mathbf{X}(x)\,\mathbf{Y}, \tag{4}$$

$$\mathbf{y}^{(k)}(x) = \mathbf{X}^{(k)}(x)\,\mathbf{Y}, \tag{5}$$

where

$$\mathbf{X}(x) = \left[\, 1 \;\; x \;\; x^2 \;\; \ldots \;\; x^N \,\right], \quad \mathbf{Y} = \left[\, y_0 \;\; y_1 \;\; y_2 \;\; \ldots \;\; y_N \,\right]^T.$$

The relation between the matrices $\mathbf{X}(x)$ and $\mathbf{X}^{(k)}(x)$ is acquired as:

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k, \tag{6}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

By substituting the relation (6) in the relation (5), we get

$$\mathbf{y}^{(k)}(x) = \mathbf{X}(x)\,\mathbf{B}^k\,\mathbf{Y}. \tag{7}$$

In addition, by putting $x \to \alpha_{kj}x + \beta_{kj}$ into the matrix relation (7), we obtain

$$\mathbf{y}^{(k)}(\alpha_{kj}x + \beta_{kj}) \cong \mathbf{y}_N^{(k)}(\alpha_{kj}x + \beta_{kj}) = \mathbf{X}(\alpha_{kj}x + \beta_{kj})\mathbf{B}^k\mathbf{Y}. \tag{8}$$

From the binomial expansion of $(\alpha_{kj}x + \beta_{kj})^N$, we can write the relation between the matrices $\mathbf{X}(\alpha_{kj}x + \beta_{kj})$ and $\mathbf{X}(x)$:

$$\mathbf{X}(\alpha_{kj}x + \beta_{kj}) = \mathbf{X}(x)\,\mathbf{B}(\alpha_{kj}, \beta_{kj}), \tag{9}$$

where

$$\mathbf{B}(\alpha_{kj},\ \beta_{kj}) = \begin{bmatrix} \binom{0}{0}\alpha_{kj}^{0}\beta_{kj}^{0} & \binom{1}{0}\alpha_{kj}^{0}\beta_{kj}^{1} & \binom{2}{0}\alpha_{kj}^{0}\beta_{kj}^{2} & \cdots & \binom{N}{0}\alpha_{kj}^{0}\beta_{kj}^{N} \\ 0 & \binom{1}{1}\alpha_{kj}^{1}\beta_{kj}^{0} & \binom{2}{1}\alpha_{kj}^{1}\beta_{kj}^{1} & \cdots & \binom{N}{1}\alpha_{kj}^{1}\beta_{kj}^{N-1} \\ 0 & 0 & \binom{2}{2}\alpha_{kj}^{2}\beta_{kj}^{0} & \cdots & \binom{N}{2}\alpha_{kj}^{2}\beta_{kj}^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}\alpha_{kj}^{N}\beta_{kj}^{0} \end{bmatrix}_{(N+1)\times(N+1)}.$$

By substituting the relation (9) into the relation (8), we reach the matrix relation:

$$\mathbf{y}^{(k)}(\alpha_{kj}x+\beta_{kj}) \cong \mathbf{y}_N^{(k)}(\alpha_{kj}x+\beta_{kj}) = \mathbf{X}(x)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^k\mathbf{Y}. \tag{10}$$

Similarly, it is clear that the matrix form of $\mathbf{y}^{(r)}(\mu_{rs}t+\gamma_{rs})$ is:

$$\mathbf{y}^{(r)}(\mu_{rs}t+\gamma_{rs}) \cong \mathbf{y}_N^{(r)}(\mu_{rs}t+\gamma_{rs}) = \mathbf{X}(t)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{Y}. \tag{11}$$

Now, we find matrix form of the Kernel function $K_{rs}(x,t)$ by means of the following procedure.

The function $K_{rs}(x,t)$ can be expressed by the truncated Taylor series as:

$$K_{rs}(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} k_{mn}^{rs}\, x^m t^n, \tag{12}$$

where

$$k_{mn}^{rs} = \frac{1}{m!\,n!}\frac{\partial^{m+n}K_{rs}(0,0)}{\partial x^m \partial t^n}, \quad m,n = 0,1,\ldots,N, \quad r = 0,1,\ldots,m_3-1.$$

Thus, the expression (12) can be written in the matrix form:

$$K_{rs}(x,t) = \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{X}^T(t), \tag{13}$$

where $\mathbf{K}_{rs} = [k_{mn}^{rs}]$, $m,n = 0,1,\ldots,N$, are the Taylor coefficients matrices of functions $K_{rs}(x,t)$ at the point (0,0).

Now, we construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, we first substitute (10), (11), and (13) into (1). After the required arrangements have been made, we obtain the matrix equation:

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^k\mathbf{Y} = f(x) + \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{X}^T(t)\mathbf{X}(t)\mathbf{B}(\mu_{rs},\gamma_{rs})\,\mathbf{B}^r\,\mathbf{Y}dt,$$

or

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\mathbf{X}(x)\mathbf{K}_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)}\mathbf{X}^T(t)\mathbf{X}(t)dt\ \mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\right\}\mathbf{Y} = f(x).$$

Following the given way for integral part, we have the matrix relation:

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\mathbf{X}(x)\mathbf{K}_{rs}\mathbf{Q}_{rs}(x)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\right\}\mathbf{Y} = f(x), \tag{14}$$

where

$$\mathbf{Q}_{rs}(x) = [q_{mn}^{rs}(x)] = \int_{u_{rs}(x)}^{v_{rs}(x)} \mathbf{X}^T(t)\mathbf{X}(t)\,\mathrm{d}t, \quad r = 0, 1, \ldots m_3,\ s = r = 0, 1, \ldots m_4,$$

$$[q_{mn}^{rs}(x)] = \frac{(v_{rs}(x))^{m+n+1} - (u_{rs}(x)_0)^{m+n+1}}{m+n+1}, \quad m, n = 0, 1, \ldots, N.$$

3 Matrix representations based on collocation points

To get an approximate solution in the form (3) of Eq. (1), we can use a matrix method based on the collocation points defined by

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N. \tag{15}$$

Now, let us substitute the collocation points (15) into Eq. (14), and thus, we obtain the system of matrix equations as:

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x_i)\mathbf{X}(x_i)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\mathbf{X}(x_i)\mathbf{K}_{rs}\mathbf{Q}_{rs}(x_i)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\right\}\mathbf{Y} = f(x_i);\quad i = 0, 1, \ldots, N,$$

or the fundamental matrix equation as:

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} \mathbf{P}_{kj}\mathbf{X}\,\mathbf{B}(\alpha_{kj},\beta_{kj})\,\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\overline{\mathbf{X}\mathbf{K}}_{rs}\overline{\mathbf{Q}}_{rs}\,\overline{\mathbf{B}}(\mu_{rs},\gamma_{rs})\overline{\mathbf{B}}^r\right\}\mathbf{Y} = \mathbf{F}, \tag{16}$$

where

$$\mathbf{P}_{kj} = \begin{bmatrix} \mathbf{P}_{kj}(t_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_{kj}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_{kj}(t_N) \end{bmatrix}_{(N+1)\times(N+1)}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \ldots & x_0^N \\ 1 & x_1 & \ldots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \ldots & x_N^N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) & 0 & \ldots & 0 \\ 0 & \mathbf{X}(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{X}(x_N) \end{bmatrix}_{(N+1)\times(N+1)^2}, \quad \bar{\mathbf{K}}_{rs} = \begin{bmatrix} \mathbf{K}_{rs} & 0 & \ldots & 0 \\ 0 & \mathbf{K}_{rs} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{K}_{rs} \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\bar{\mathbf{Q}}_{rs} = \begin{bmatrix} \mathbf{Q}_{rs} & 0 & \dots & 0 \\ 0 & \mathbf{Q}_{rs} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{Q}_{rs} \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\bar{\mathbf{B}}(\mu_{rs},\gamma_{rs}) = \begin{bmatrix} \mathbf{B}(\mu_{rs},\gamma_{rs}) & 0 & \dots & 0 \\ 0 & \mathbf{B}(\mu_{rs},\gamma_{rs}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}(\mu_{rs},\gamma_{rs}) \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\overline{\mathbf{B}}^{r} = \begin{bmatrix} \mathbf{B}^{r} \\ \mathbf{B}^{r} \\ \vdots \\ \mathbf{B}^{r} \end{bmatrix}_{(N+1)^2\times(N+1)},$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}(t_0) \\ \mathbf{f}(t_1) \\ \vdots \\ \mathbf{f}(t_N) \end{bmatrix}_{(N+1)\times 1}$$

The main matrix Eq. (16) corresponds to a system of $N+1$ algebraic equations for the $N+1$ unknown Taylor coefficients $y_0, y_1, \dots, y_N$. We can write it briefly in the following form:

$$\mathbf{WY} = \mathbf{F} \quad \text{or}\,[\mathbf{W};\mathbf{F}], \tag{17}$$

where

$$\mathbf{W} = [w_{pq}] = \sum_{k=0}^{m_1}\sum_{j=0}^{m_2} \mathbf{P}_{kj}\mathbf{X}\,\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{B}^{k} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs}\overline{\mathbf{XK}}_{rs}\overline{\mathbf{Q}}_{rs}\,\overline{\mathbf{B}}(\mu_{rs},\gamma_{rs})\overline{\mathbf{B}}^{r}.$$

On the other hand, we get the matrix form of the mixed conditions (2) by means of the relation (7) as:

$$\mathbf{U}_i\mathbf{Y} = \eta_i \Rightarrow [\mathbf{U}_i;\eta_i], \quad i = 0,1,\dots,m_1-1, \tag{18}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m_1-1} [a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b)] \quad \mathbf{B}^{k} = [\,u_{i0}\;u_{i1}\;\dots\;u_{iN}\,].$$

To find the Taylor polynomial solution of Eq. (1) under the mixed conditions (2), we replace row matrix (18) by any m_1 rows of (17) and we get the augmented matrix as:

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]=\left[\begin{array}{cccc;{2pt/2pt}c} w_{00} & w_{01} & \dots & w_{0N} & f(x_0) \\ w_{10} & w_{11} & \dots & w_{1N} & f(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-m_1,0} & w_{N-m_1,1} & \dots & w_{N-m_1,N} & f(x_{N-m_1}) \\ u_{00} & u_{01} & \dots & u_{0N} & \eta_0 \\ u_{10} & u_{11} & \dots & u_{1N} & \eta_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m_1-1,0} & u_{m_1-1,1} & \dots & u_{m_1-1,N} & \eta_{m_1-1} \end{array}\right].$$

If rank $\widetilde{\mathbf{W}}=\operatorname{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]=N+1$, then we can write $\mathbf{Y}=\widetilde{\mathbf{W}}^{-1}\widetilde{\mathbf{F}}$. Hence, the matrix $\mathbf{Y}$ and also the Taylor coefficients y_n, $n=0,1,\dots,N$ are uniquely determined. Therefore, we get the demanded Taylor polynomial solution:

$$y_N(x)=\mathbf{X}(x)\mathbf{Y}. \tag{19}$$

If rank $\widetilde{\mathbf{W}}=\operatorname{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]<N+1$, we find infinite solution depending on the parameter. Otherwise, if rank $\widetilde{\mathbf{W}}\neq\operatorname{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]$, then there is not a solution.

4 Residual correction and error estimation

In this section, accuracy of the approximate solutions is checked by substituting the solutions into Eq. (1):

$$E_N(x)$$
$$=\left|\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}P_{kj}(x)\,y_N^{(k)}(\alpha_{kj}x+\beta_{kj})-f(x)-\sum_{r=0}^{m_3}\sum_{r=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)}K_{rs}(x,t)\,y_N^{(r)}(\mu_{rs}x+\gamma_{rs})\mathrm{d}t\right|.$$

We expect that $E_N(x)=0$ on the collocation points. The closer $y(x)\cong y_N(x)$ the closer $E_N(x)\cong 0$. Accuracy of the approximate solutions may not give any information about the absolute errors. To remove this limitation, we can apply the residual correction procedure to estimate the absolute errors (Oliveira 1980; Çelik 2005; Shahmorad 2005).

Now, we give an error estimation based on the residual function for Taylor collocation method. Using this procedure, it can be estimated the optimal M giving minimal absolute error. For modifying the procedure to Eq. (1), first, we get the residual function for Taylor polynomial solution (19) as:

$$\begin{aligned}R_N(x)=&\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}P_{kj}(x)\,y_N^{(k)}(\alpha_{kj}x+\beta_{kj})\\&-\left(f(x)+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)}K_{rs}(x,t)\,y_N^{(r)}(\mu_{rs}x+\gamma_{rs})\mathrm{d}t\right),\end{aligned} \tag{20}$$

where $y_N(x)$ denotes the approximate solution (19). By adding (20) into the both side of Eq. (1), we have

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)\, e_N^{(k)}(\alpha_{kj}x+\beta_{kj}) - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\, e_N^{(r)}(\mu_{rs}x+\gamma_{rs})\mathrm{d}t = -R_N, \tag{21}$$

where $e_N(x) = y(x) - y_N(x)$.

Let $e_{N,M}(x)$ be the Taylor series solution of (21). If

$$\left\| e_N(x) - e_{N,M}(x) \right\| < \varepsilon,$$

are sufficiently small, then the absolute error can be estimated by $e_{N,M}(x)$. Hence, the optimal M for the absolute errors can be obtained measuring the error functions $e_{N,M}(x)$ for different M values in any norm.

5 Numerical experiments

In this section, some examples are given to explain the procedure with details and demonstrate the effectiveness of the method. All computations and graphs are performed by codes written in Maple and Matlab.

Example 1 Let us first consider the first order pantograph-type Volterra integro-differential equation:

$$xy(x) + 2y(2x+1) + y'(x-1) - xy'(x) - x^2y''(x) + \int_x^{x+2} xty(t)\mathrm{d}t + \int_{\frac{x}{2}}^{x} (x+t)y'(t+1)\mathrm{d}t$$

$$+ \int_0^{2x-1} (x-t)y'(t)\mathrm{d}t = 2x^4 + 7x^3 + \frac{79}{4}x^2 + \frac{8}{3}, \tag{22}$$

with initial conditions

$$y(0) = 1 \text{ and } y'(0) = 0, \tag{23}$$

where $P_{00}(x) = x$, $P_{01}(x) = 2$, $P_{12}(x) = 1$, $P_{13}(x) = -x$, $P_{24}(x) = -x^2$, $K_{00}(x,t) = xt$, $K_{11}(x,t) = x+t$, $K_{12}(x,t) = x-t$ and $f(x) = 2x^4+7x^3+\frac{79}{4}x^2+\frac{8}{3}$. We seek the approximate solution of the problem using the truncated Taylor series (3) for $N = 2$:

$$y(x) \cong y_2(x) = \sum_{n=0}^{2} y_n\, x^n. \tag{24}$$

Now, we determine the collocation points (15) for $N = 2$ in [0,1]. Then,

$$x_0 = 0,\ \ x_1 = \frac{1}{2},\ \ x_2 = 1.$$

The main matrix equation of Eq. (1) is written using (16) as:

$$\left\{ \begin{array}{l} \mathbf{P}_{00}\mathbf{X} + \mathbf{P}_{01}\mathbf{X}\,\mathbf{B}(\alpha_{01},\beta_{01}) + \mathbf{P}_{12}\mathbf{X}\,\mathbf{B}(\alpha_{12},\beta_{12})\mathbf{B} + \mathbf{P}_{13}\mathbf{X}\,\mathbf{B} + \mathbf{P}_{24}\mathbf{X}\,\mathbf{B}^2 + \lambda_{00}\overline{\mathbf{X}\mathbf{K}}_{00}\overline{\mathbf{Q}}_{00} \\ + \lambda_{11}\overline{\mathbf{X}\mathbf{K}}_{11}\overline{\mathbf{Q}}_{11}\,\overline{\mathbf{B}}(\mu_{11},\gamma_{11})\overline{\mathbf{B}} + \lambda_{12}\overline{\mathbf{X}\mathbf{K}}_{12}\overline{\mathbf{Q}}_{12}\,\overline{\mathbf{B}} \end{array} \right\} \mathbf{Y} = \mathbf{F},$$

where $\alpha_{01} = 2$, $\beta_{01} = 1$, $\alpha_{12} = 1$, $\beta_{12} = -1$, $\lambda_{00} = \lambda_{11} = \lambda_{12} = 1$, and $\mu_{11} = \gamma_{11} = 1$. In addition, we can write it briefly,

$$[\mathbf{W};\ \mathbf{F}] = \begin{bmatrix} 2 & 5/2 & 2/3 & ; & 8/3 \\ 4 & 725/96 & 557/48 & ; & 749/48 \\ 7 & 409/24 & 461/12 & ; & 545/12 \end{bmatrix}.$$

The matrix forms of initial conditions are calculated as:

$$\mathbf{U}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{and}\, \mathbf{U}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Then, the new augmented matrix can be found by adding the augmented matrix form of the initial conditions into the last rows of the augmented matrix $[\mathbf{W};\ \mathbf{F}]$, above and from (23):

$$[\widetilde{\mathbf{W}};\ \widetilde{\mathbf{F}}] = \begin{bmatrix} 2 & 5/2 & 2/3 & ; & 8/3 \\ 1 & 0 & 0 & ; & 1 \\ 0 & 1 & 0 & ; & 0 \end{bmatrix}.$$

By solving the system of corresponding augmented matrix $[\widetilde{\mathbf{W}};\ \widetilde{\mathbf{F}}]$, the Taylor coefficients are uniquely determined as:

$$y_0 = 1, \quad y_1 = 0, \quad y_2 = 1.$$

Finally, the determined coefficients are substituted into Eq. (24) and the approximate solution is obtained as $y_2(x) = x^2 + 1$ which is the exact solution of (22) and (23).

Example 2 Now, we consider Volterra delay integro-differential equation, Çelik (2006):

$$y'(x) = y(x-1) + \int_{x-1}^{x} y(t)\mathrm{d}t, \tag{25}$$

with the initial condition

$$y(0) = 1, \tag{26}$$

and the exact solution is $y(x) = e^x$. Let us first write (25) in the form:

Table 1 Comparison of the exact solution and Taylor polynomial solutions for different N values in Example 2

x	Exact solution	Taylor polynomial solutions		
		$N=4$	$N=6$	$N=9$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	1.105171	1.104943	1.105166	1.105171
0.2	1.221403	1.220837	1.221400	1.221403
0.3	1.349859	1.348912	1.349864	1.349860
0.4	1.491825	1.490522	1.491841	1.491825
0.5	1.648721	1.647139	1.648749	1.648722
0.6	1.822119	1.820355	1.822156	1.822119
0.7	2.013753	2.011883	2.013798	2.013753
0.8	2.225541	2.223556	2.225593	2.225542
0.9	2.459603	2.457327	2.459659	2.459604
1.0	2.718282	2.715270	2.718338	2.718283

Table 2 Comparison of the absolute errors of Example 2

x	Absolute errors		
	$E_N = E_4$	$E_N = E_6$	$E_N = E_9$
0.0	0.000000	0.000000	0.000000
0.1	0.227729E−3	0.483575E−5	0.264391E−7
0.2	0.566199E−3	0.251246E−5	0.109016E−6
0.3	0.946710E−3	0.520238E−5	0.223732E−6
0.4	0.130271E−2	0.159204E−4	0.347268E−6
0.5	0.158266E−2	0.272853E−4	0.461544E−6
0.6	0.176427E−2	0.375203E−4	0.556043E−6
0.7	0.187019E−2	0.457277E−4	0.627980E−6
0.8	0.198540E−2	0.517840E−4	0.680899E−6
0.9	0.227639E−2	0.556557E−4	0.722424E−6
1.0	0.301238E−2	0.559455E−4	0.761573E−6

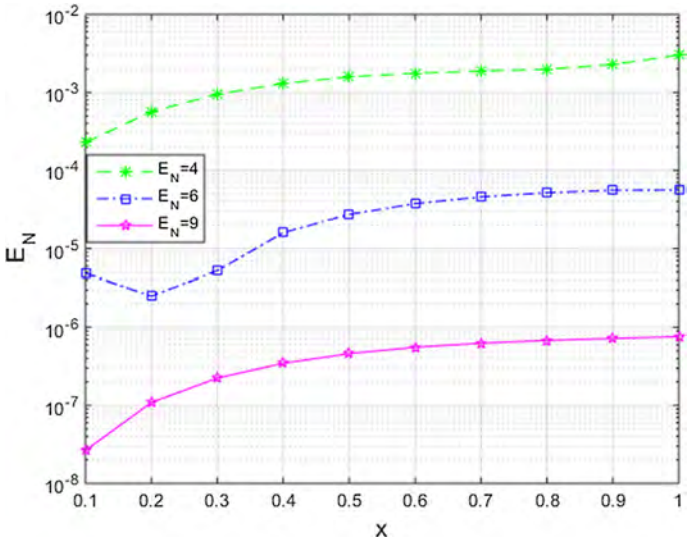

Fig. 1 Logarithmic plot for the comparison of the absolute errors for Example 2

$$y'(x) = y(x-1) - \int_0^{x-1} y(t)\mathrm{d}t + \int_0^{x} y(t)\mathrm{d}t.$$

Similarly, we solve the problem using the same procedure in Example 1. Then, we have the approximate solutions for different N values which can be seen in Table 1. We have comparison of the absolute errors in Table 2 and Fig. 1. Moreover, we can see the comparison of $E_{N,M}$ for different N, M values in Table 3. It is clearly seen that we have the appropriate solutions and smaller values when N and M values are increasing.

Table 3 Comparison of the $E_{N,M}$ residual error functions of Example 2

x	$N, M = 4, 5$	$N = 6, 7$	$N = 8,9$
0.0	0.000000	0.000000	0.000000
0.1	0.273242E−26	0.157863E−28	0.412545E−30
0.2	0.440304E−26	0.650871E−28	0.828376E−30
0.3	0.497880E−26	0.133911E−27	0.116943E−29
0.4	0.465618E−26	0.208281E−27	0.137195E−29
0.5	0.376296E−26	0.277078E−27	0.139704E−29
0.6	0.265702E−26	0.333808E−27	0.124120E−29
0.7	0.164521E−26	0.377016E−27	0.946937E−30
0.8	0.936380E−27	0.409212E−27	0.613290E−30
0.9	0.629600E−27	0.434281E−27	0.406395E−30
1.0	0.721235E−27	0.453476E−27	0.570047E−30

Example 3 We reach the approximate solution of the second-order pantograph VIDE of the neutral type:

$$y''(x) = (x+1)y'(x) - y(x) + \int_{-1}^{x} [xy(t) + y'(t) + ty''(t)]\mathrm{d}t + g(x),$$

where $g(x) = (x+1))(\sin(x) - \sin(1))$, Reutskiy (2016). Initial conditions are $y(-1)) = \cos(1))$ and $y'(-1) = \sin(1)$ and the exact solution is $y(x) = \cos(x)$ (Tables 4, 5; Fig. 2).

Table 4 Comparison of the absolute errors of Example 3

x	Absolute errors		
	$E_N = E_4$	$E_N = E_6$	$E_N = E_8$
0.0	3.62515E−3	6.21567E−4	3.90818E−5
0.1	3.97469E−3	6.93748E−4	4.33341E−5
0.2	4.32996E−3	7.68123E−4	4.76917E−5
0.3	4.69820E−3	8.46325E−4	5.22495E−5
0.4	5.08831E−3	9.30535E−4	5.71343E−5
0.5	5.51328E−3	1.02372E−4	6.25188E−5
0.6	5.99365E−3	1.12998E−4	6.86417E−5
0.7	6.56179E−3	1.25507E−4	7.58365E−5
0.8	7.26717E−3	1.40706E−3	8.45759E−5
0.9	8.18240E−3	1.59733E−3	9.55370E−5
1.0	9.40910E−3	1.84175E−3	1.09610E−4

Table 5 Comparison of the $E_{\max}$ maximum absolute errors for different methods and different N, M values in Example 3

N, M	$E_{\max}$ absolute errors		
	Taylor collocation method	The backward substitution method (in Reutskiy et al. 2016)	Legendre spectral collocation method (in Wei and Chen 2014)
(2, 3)	2.1E−3	5.3E−2	7.4E−3
(4, 5)	3.6E−4	7.0E−3	6.2E−5
(6, 7)	1.3E−5	1.3E−5	2.8E−7
(8, 9)	4.7E−7	1.3E−8	7.7E−10

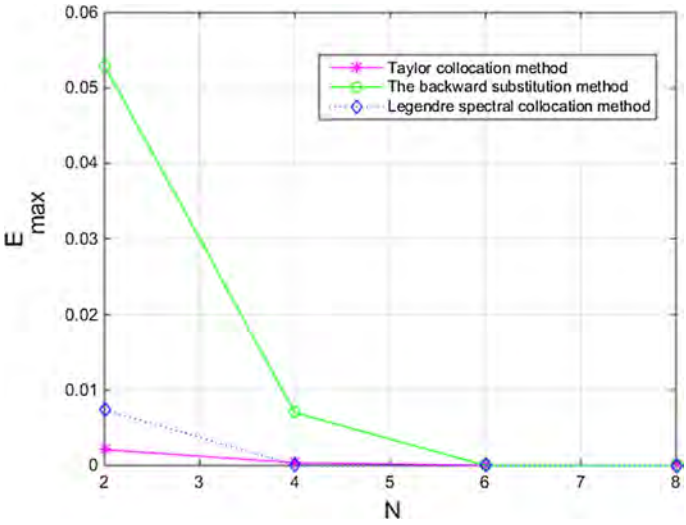

Fig. 2 Comparison of the maximum absolute errors for $N = 2, 4, 6, 8$ in Example 3

Example 4 We consider the Volterra delay integro-differential equation of partially variable coefficients:

$$y'(x) + (6 + \sin(x))y(x) - y\left(x - \frac{\pi}{4}\right) = 5e^{\cos(x)} + \int_{x-\frac{\pi}{4}}^{x} \sin(t)y(t)dt, \quad x \geq 0.$$

Initial condition is given as $y(0) = e$ and the exact solution is $y(x) = e^{\cos(x)}$ (Rihan et al. 2009) (Tables 6, 7, 8; Fig. 3).

Table 6 Comparison of the absolute errors of Example 4

x	Absolute errors		
	$E_N = E_4$	$E_N = E_9$	$E_N = E_{13}$
0.0	0.000000	0.000000	0.000000
0.1	0.63085E−3	0.12075E−4	0.67916E−5
0.2	0.52753E−3	0.40761E−6	0.89207E−5
0.3	0.35634E−4	0.20123E−4	0.73098E−5
0.4	0.62596E−3	0.34247E−4	0.39875E−5
0.5	0.83332E−3	0.37675E−4	0.84630E−6
0.6	0.49358E−3	0.31869E−4	0.10439E−5
0.7	0.69134E−4	0.21541E−4	0.15211E−5
0.8	0.19473E−3	0.11447E−4	0.10260E−5
0.9	0.32717E−2	0.45084E−5	0.18551E−6
1.0	0.12256E−1	0.11688E−5	0.50482E−6

Table 7 CPU times for $N = 4$, 9 and 13 of Example 4

Wall clock time (s)		
$N = 4$	$N = 9$	$N = 13$
54.24	60.30	67.8

Table 8 Comparison of the $E_{N,M}$ residual error functions of Example 4

x	$N, M = 4, 5$	$N = 9, 10$	$N = 13, 14$
0.0	0.000000	0.000000	0.000000
0.1	0.13249E−6	0.18564E−10	0.77883E−15
0.2	0.19423E−6	0.25852E−10	0.30750E−15
0.3	0.18064E−6	0.25877E−10	0.19760E−14
0.4	0.11481E−6	0.32028E−10	0.38264E−14
0.5	0.43419E−7	0.67494E−10	0.73509E−14
0.6	0.32789E−7	0.16568E−9	0.16943E−13
0.7	0.16487E−6	0.37063E−9	0.40908E−13
0.8	0.53322E−6	0.73745E−9	0.92470E−13
0.9	0.12391E−5	0.13327E−8	0.19078E−12
1.0	0.23872E−5	0.22349E−8	0.36194E−12

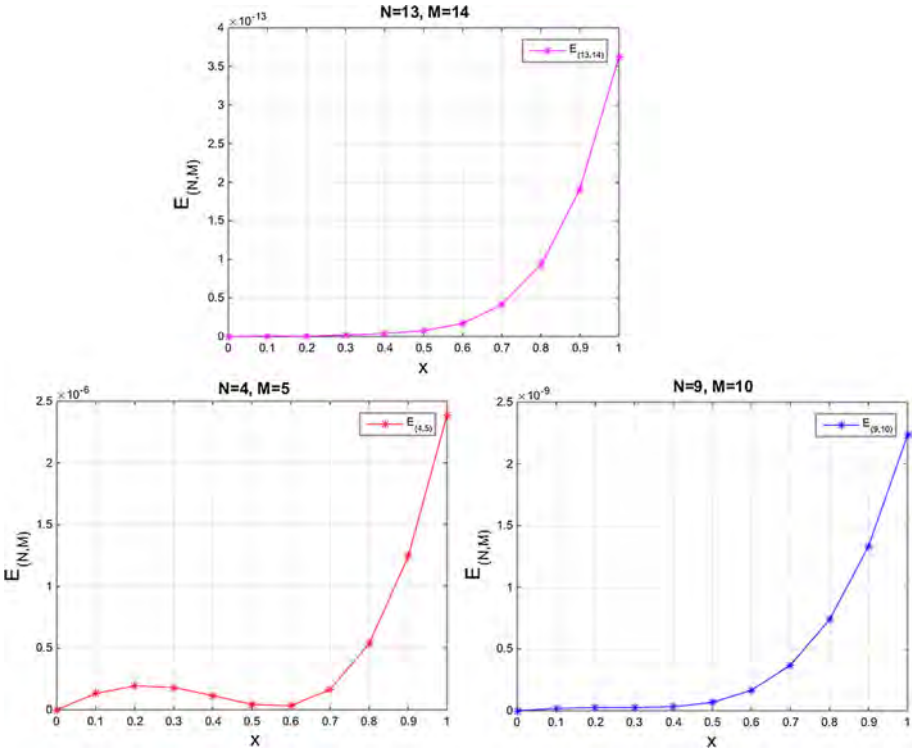

Fig. 3 Comparison of $E_{13,14}$, $E_{4,5}$, and $E_{9,10}$ residual error functions of Example 4

6 Conclusion

This paper has presented a numerical method to solve functional delay integro-differential equations with variable bounds. The method is based on the truncated Taylor series expansion. The approximate solutions can be found very close to the exact solutions when N is chosen large enough. In addition, tables and figures have been shown that the error decreases when N and M increase. In addition, the results have been compared with the data of any other methods and validity of the method has been approved. Furthermore, CPU times have been given to show the efficiency of the method.

Moreover, the residual error function has been presented which helps us for finding satisfactory results. An important advantage of the method is that the Taylor polynomial coefficients of the solution can be found very easily using the computer programs: Maple and Matlab. These are significant advantages compared to the majority of the existing methods (Wei and Chen 2014; Maleknejad and Mahmoudi 2003).

As a result, the technique can be applied on particular type of mathematical models, but some modifications are required.

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