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FOUNDATIONS

The concept of σ-algebraic soft set

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Abstract In this paper, the concept of σ-algebraic soft set which can be used in decision-making process is introduced and some of its structural properties are studied. In order to compare the parameters in soft set theory, we give several characterizations using measurement on the initial universe. Then its applications are given.

Keywords Soft set · σ-Algebra · Measurable set

1 Introduction

In science, engineering, economics and environmental sciences, many scientists seeks to develop a mathematical model to analyze the uncertainty. But we cannot successfully use classical mathematical methods for those models. Firstly, Zadeh (1965) proposed fuzzy set theory which is an important tool to solve problems that contains vagueness. This theory has been studied by many scientists over the years. However, accurate, permanent and healthy solution of encountered problems could only be done with the right parameterization in real life or applied sciences. The most straightforward and easy mathematical structure that allows it, of course, is the theory of soft sets which is defined by Molodtsov (1999) in 1999. He established the fundamental results of this theory. He applied this theory in analysis, game theory and probability theory. In Maji et al. (2003), Ali et al. (2009), Kharal and Ahmad (2009), Babitha and Sunil (2010), Min (2012), set-theoretical operations of this theory such as subset, union, intersection, mappings and relations have been defined and studied. In Pei and Miao (2005), showed that every soft set over an initial universe is an information system. In Aktaş and Çağman (2007), Feng et al. (2008), soft algebraic structures on given initial universe are described. Soft topology has been defined by Shabir and Naz (2011). They defined some fundamental structures in soft topological spaces. Because soft set theory is a parameterization of subsets of a given universe, choosing the appropriate parameters related to problem is very important for solving the problem in the problem universe. So then, Chen et al. (2005) gave some reduction technique to solve relevant problem for stack of parameters, i.e., they gave a reduction method to determine the parameters that are important for problem and they proposed decision-making method with this reduction. But even in this case, we do not know which parameters would be more appropriate, i.e., which parameters are more preferable to choosing. Comparison among the parameters directly affects the decision-making process. Therefore, a comparison is necessary for interested parameters.

In this paper, to cope with this problem we define the concept of σ-algebraic soft set using the concept of measurement on an initial universe. Toward the end of the paper, we give some characterizations to compare parameters such as *preferability*, *indiscernibility*, *weight of a parameter* and *impact of a parameter*. Besides, we showed that a function which is called *parametric weight of a soft set* is a measure on all σ-algebraic soft sets over any initial given universe. Finally, we give a result to compare the parameters of any soft set given over the initial universe U among themselves.

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2 Preliminaries

2.1 Soft set theory

Throughout this paper U will be an initial universe, E will be the set of all possible parameters which are attributes, characteristic or properties of the objects in U, and the set of all subsets of U will be denoted by $\mathcal{P}(U)$.

Definition 2.1 (Molodtsov 1999) Let A be a subset of E. A pair (F, A) is called a *soft set* over U where $F : A \to \mathcal{P}(U)$ is a set-valued function.

As mentioned in Maji et al. (2003), a soft set (F, A) can be viewed $(F, A) = \{a = F(a) \mid a \in A\}$ where the symbol "$a = F(a)$" indicates that the approximation for $a \in A$ is $F(a)$.

Definition 2.2 (Pei and Miao 2005) For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B) and is denoted by $(F, A)\widetilde{\subseteq}(G, B)$ if

(i) $A \subset B$ and,
(ii) $\forall a \in A$, $F(a) \subset G(a)$.

Definition 2.3 (Pei and Miao 2005) Two soft sets (F, A) and (G, B) over a common universe U are said *soft equal* if (F, A) is a soft subset of (G, B), and (G, B) is a soft subset of (F, A).

Definition 2.4 (Ali et al. 2009) Let U be an initial universe set, E be the universe set of parameters, and $A \subset E$.

(i) (F, A) is called a *relative null soft set* (with respect to the parameter set A), denoted by Φ_A, if $F(a) = \varnothing$ for all $a \in A$.
(ii) (F, A) called a *relative whole soft set* (with respect to the parameter set A), denoted by $\mathcal{U}_A$, if $F(a) = U$ for all $a \in A$.

The relative whole soft set $\mathcal{U}_E$ with respect to the universe set of parameters E is called *the absolute soft set* over U.

Let U be an initial universe, E be a parameters set. The family of all soft sets over U via E is denoted by $\mathcal{S}(U; E)$. Moreover, the family of soft subsets of a given soft set (F, A) is denoted by $\mathcal{P}(F, A)$ like as power set of a set.

Definition 2.5 (Pei and Miao 2005) Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \varnothing$. The *intersection*[1] of (F, A) and (G, B) is denoted by $(F, A)\widetilde{\cap}(G, B)$, and is defined as $(F, A)\widetilde{\cap}(G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

We will use this definition of intersection given in Pei and Miao (2005) instead of the one given in Maji et al. (2003), because generally $F(c)$ and $G(c)$ are not necessarily equal for $c \in C$. So this definition is more applicable to soft sets.

Definition 2.6 Let (F, A) and (G, B) be soft sets over U.(F, A) and (G, B) are called *disjoint soft sets* if $F(a) \cap G(b) = \varnothing$ for all $a \in A$, $b \in B$.

Note that, if (F, A) and (G, B) are disjoint then it can be easily seen that $(F, A)\widetilde{\cap}(G, B) = \Phi$.

Definition 2.7 (Maji et al. 2003) The *union* of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) , denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$, where $C = A \cup B$, and $\forall c \in C$,

$$H(c) = \begin{cases} F(c) & , \text{if } c \in A - B \\ G(c) & , \text{if } c \in B - A \\ F(c) \cup G(c) & , \text{if } c \in A \cap B \end{cases}$$

Definition 2.8 (Maji et al. 2003) Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) **AND** (G, B) denoted by $(F, A) \wedge (G, B)$ and is defined by $(F, A) \wedge (G, B) = (H, A \times B)$ where $H((a, b)) = F(a) \cap G(b)$, for all $(a, b) \in A \times B$.

Definition 2.9 (Maji et al. 2003) Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) **OR** (G, B) denoted by $(F, A) \vee (G, B)$ and is defined by $(F, A) \vee (G, B) = (H, A \times B)$ where $H((a, b)) = F(a) \cup G(b)$, for all $(a, b) \in A \times B$.

Definition 2.10 (Pei and Miao 2005) The *complement*[2] of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.

Example 2.11 Let $U = \{a, b, c\}$ be universe, $E = \{1, 2, 3\}$ be parameter set and $A = \{1, 3\} \subset E$. From Definition 2.1, $(F, A) = \{1 = \{a, b\},\ 3 = \{b, c\}\}$ is a soft set over U.

Definition 2.12 (Babitha and Sunil 2010) Let (F, A) and (G, B) be two soft set over U, then the *cartesian product* of (F, A) and (G, B) is defined as, $(F, A) \times (G, B) = (H, A \times B)$, where $H : A \times B \to \mathcal{P}(U \times U)$ and $H(a, b) = F(a) \times G(b)$, where $(a, b) \in A \times B$.

In addition to these, we can define the soft function that given function between universes and parameters sets.

Kharal and Ahmad (2009) defined the concept of soft function as the follows. We have modified appropriately.

[1] Note that intersection is also known as bi-intersection in Feng et al. (2008) and as restricted intersection in Ali et al. (2009)

[2] Note that complement is known as relative complement in Ali et al. (2009)

Definition 2.13 (Soft Function) (Kharal and Ahmad 2009) Let U_1, U_2 be initial universes, E_1, E_2 be parameters sets, φ be a function from U_1 to U_2 and ψ be a function from E_1 to E_2. Then the pair (φ, ψ) is called *soft function* from $S(U_1, E_1)$ to $S(U_2, E_2)$. The *image* of each $(F, A) \in S(U_1, E_1)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)(F, A) = (\varphi F, \psi(A))$ and is defined as following;

$$(\varphi F)(\beta) = \begin{cases} \varphi\left(\bigcup_{\alpha \in \psi^{-1}(\beta) \cap A} F(\alpha)\right), & \psi^{-1}(\beta) \cap A \neq \varnothing \\ \varnothing, & \text{otherwise} \end{cases}$$

for each $\beta \in \psi(A)$.

Similarly, the *inverse image* of each $(G, B) \in S(U_2, E_2)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)^{-1}(G, B) = (\varphi^{-1}G, \psi^{-1}(B))$ and is defined as following;

$$(\varphi^{-1}G)(\alpha) = \begin{cases} \varphi^{-1}(G(\psi(\alpha))) & , \psi(\alpha) \in B \\ \varnothing & , \text{otherwise} \end{cases}$$

for each $\alpha \in \psi^{-1}(B)$.

Min described the similarity in soft set theory and gave some results in Min (2012). He gave the definition of similarity between two soft sets as follows.

Definition 2.14 (Min 2012) Let (F, A) and (G, B) be soft sets over a common universe set U. Then (F, A) is *similar* to (G, B) (simply $(F, A) \cong (G, B)$) if there exists a bijective function $\phi : A \to B$ such that $F(\alpha) = (G \circ \phi)(\alpha)$ for every $\alpha \in A$, where $(G \circ \phi)(\alpha) = G(\phi(\alpha))$.

Now, we can give the definition of generalized form of similarity between soft sets over different universes as follows:

Definition 2.15 Let E be a set of parameters, U and V be two universes and (F, A) and (G, B) be soft sets over U and V, respectively, where $A, B \subseteq E$. We called that (F, A) *similar* to (G, B) if there exist bijective functions $f : U \to V$ and $\phi : A \to B$ such that $(f \circ F)(\alpha) = (G \circ \phi)(\alpha)$ for every $\alpha \in A$.

Note that, the given functions in the above definition should not be confused with the soft functions.

Definition 2.16 (Li et al. 2013) Let (F, A) be a soft set over U. (F, A) is called *topological* if $\{F(e) \mid e \in A\}$ is a topology on U.

Therewithal, in Min (2014), Min defined the concept of open soft set over any topological universe which is a topological space as follows.

Definition 2.17 (Min 2014) Let $(U, \mathcal{O})$ be a topological universe, (F, A) be a soft set over U where $A \subseteq E$. (F, A) is called an *open soft set* if $F(e)$ is open in U, i.e., $F(e) \in \mathcal{O}$ for all $e \in A$.

2.2 σ-algebras, measurable functions, measures

As known, σ-algebra plays the key role in the measure theory. We recall basic properties of σ-algebras and measure.

Definition 2.18 (Emelyanov 2007) A collection $\mathcal{A}$ of subsets of a set U is called a *σ-algebra* if

(a) $U \in \mathcal{A}$,
(b) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$,
(c) given a sequence $(A_i)_{i \in I} \subseteq \mathcal{A}$, we have $\bigcup_{i \in I} A_i \in \mathcal{A}$.

If $\mathcal{A}$ is a σ-algebra on U, then we obtain the following lemma.

Lemma 2.19 *(Emelyanov 2007)*

(1) $\varnothing \in \mathcal{A}$,
(2) if $A_i \in \mathcal{A}$ *for* $i = 1, 2, \ldots, n$ *then* $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$,
(3) if $A_i \in \mathcal{A}$ *for* $i \in \mathbb{N}$, *then* $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$,
(4) $A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$.

Proposition 2.20 *(Emelyanov 2007) Let* $\{\mathcal{A}_i\}_{i \in I}$ *be a non-empty family of σ-algebras in* $\mathcal{P}(U)$, *then* $\mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i$ *is also a σ-algebra.*

Definition 2.21 (Emelyanov 2007) Let $\mathcal{G} \subseteq \mathcal{P}(U)$, then the set of all σ-algebras containing $\mathcal{G}$ is non-empty since it contains $\mathcal{P}(U)$. The smallest σ-algebra which is containing $\mathcal{G}$ is called the *σ-algebra generated by* $\mathcal{G}$ and denoted by $\sigma(\mathcal{G})$.

An important special case of this notion is the following.

Definition 2.22 (Emelyanov 2007) Let U be a topological space and $\mathcal{O}$ be the family of all open subsets of U. The σ-algebra generated by $\mathcal{O}$ is called *Borel algebra* of U and denoted by $\mathfrak{B}(U)$.

Definition 2.23 (Halmos 1950) Let U, V be non-empty sets, $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras on U and V, respectively. The σ-algebra for the corresponding product space $U \times V$ is called *product σ-algebra* and is defined by

$$\mathcal{A} \times \mathcal{B} = \sigma\left(\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}\right).$$

Definition of the notion of measure which is important tool in mathematical analysis is below.

Definition 2.24 (Emelyanov 2007) Let $\mathcal{A}$ be a σ-algebra. A function $\mu : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ is called a *measure*, if

(1) $\mu(\varnothing) = 0$,
(2) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$,

(3) $\mu(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mu(A_i)$ for any sequence $(A_i)_{i\in I}$ of pairwise disjoint sets from $\mathcal{A}$, that is $A_i \cap A_j = \varnothing$ for $i \neq j$. The axiom of (3) is called *σ-additivity* of the measure μ. As usual, we will also assume that any measure under consideration satisfies the following axiom:

(4) for any subset $A \in \mathcal{A}$ with $\mu(A) = \infty$, there exists $B \in \mathcal{A}$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$.

Definition 2.25 (Emelyanov 2007) Let $\mathcal{A}$ be a σ-algebra on U and μ be a measure on $\mathcal{A}$. Then the triple $(U, \mathcal{A}, \mu)$ is called a *measure space*. The sets belonging to $\mathcal{A}$ are called *measurable sets*.

Definition 2.26 (Emelyanov 2007) A measure space $(U, \mathcal{A}, \mu)$ is called *σ-finite* if there is a sequence $(A_i)_{i=1}^{\infty}$, $A_i \in \mathcal{A}$, satisfying $U = \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$ for all i.

Definition 2.27 (Emelyanov 2007) A measure space $(U, \mathcal{A}, \mu)$ is called *finite* if $\mu(U) < \infty$. In particular, if $\mu(U) = 1$, then the measure space is said to be *probabilistic*, and μ is said to be a *probability*.

Definition 2.28 (Emelyanov 2007) Let $(U, \mathcal{A}_U, \mu_U)$ and $(V, \mathcal{A}_V, \mu_V)$ be two measurable spaces and $f : U \to V$ be a function. We call that f is a *measurable function* if $f^{-1}[A] \in \mathcal{A}_U$ for each $A \in \mathcal{A}_V$.

3 σ-algebraic soft sets

In this section, we have introduced the notion of σ-algebraic soft set and investigated its structural properties. Now, we define the σ-algebraic soft set as follows.

Definition 3.1 Let U be a universe and E be a set of parameters, (F, A) be a soft set over U where $A \subseteq E$ and $\mathcal{A}$ be a σ-algebra on U. We called that (F, A) is a *σ-algebraic soft set* over U if $F(e) \in \mathcal{A}$ for all $e \in A$.

The family of all σ-algebraic soft set over U via E is denoted by $\sigma\mathcal{S}(U; E)$.

Example 3.2 Let U be a universe. Since $\mathcal{P}(U)$ is a σ-algebra on U, all soft sets over U is σ-algebraic.

Example 3.3 Let $U = \mathbb{N}$,

$$\mathcal{A}=\{\varnothing, \{1, 3, 5, \ldots, 2n-1, \ldots\}, \{2, 4, 6, \ldots, 2n, \ldots\}, \mathbb{N}\}$$

is a σ-algebra on U. $E = \{a, b, c, d\}$ and the soft set

$$(F, A) = \{a = \{1, 3, 5, \ldots, 2n - 1, \ldots\},\ d = \{2, 4, 6, \ldots, 2n, \ldots\}\}$$

over U is a σ-algebraic soft set where $A = \{a, d\} \subseteq E$.

In Zhu and Wen (2010), defined a probabilistic soft set over a given universe. Note that, every probabilistic soft set over any universe is a σ-algebraic soft set.

Theorem 3.4 *Relative null and absolute soft sets over a universe are σ-algebraic soft sets.*

Proof It is clear from Definitions 2.18 and 3.1.

Theorem 3.5 *Let (F, A) and (G, B) be a σ-algebraic soft set over U. Then $(F, A)\tilde{\cap}(G, B)$ is also σ-algebraic soft set over U.*

Proof Let's say $(H, C) = (F, A)\tilde{\cap}(G, B)$. So, $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for each $e \in C$. Since (F, A) and (G, B) are σ-algebraic and from Lemma 2.19 (2), $H(e) \in \mathcal{A}$ for each $e \in C$. Hence (H, C) is a σ-algebraic soft set over U.

Theorem 3.6 *Let (F, A) and (G, B) be a σ-algebraic soft set over U. Then $(F, A)\tilde{\cup}(G, B)$ is also σ-algebraic soft set over U.*

Proof Let be $(H, C) = (F, A)\tilde{\cup}(G, B)$. Then $C = A \cup B$ and $H(e) = F(e)$ if $e \in A - B$, $H(e) = G(e)$ if $e \in B - A$ and $H(e) = F(e) \cup G(e)$ if $e \in A \cap B$. Since (F, A) and (G, B) are σ-algebraic soft sets and from definition of σ-algebra, we obtain that (H, C) is a σ-algebraic soft set over U.

Theorem 3.7 *If (F, A) and (G, B) are σ-algebraic soft sets over U, then $(F, A) \wedge (G, B)$ is also σ-algebraic soft set over U.*

Proof Similar to proof of Theorem 3.4. □

Theorem 3.8 *If (F, A) and (G, B) are σ-algebraic soft sets over U, then $(F, A) \vee (G, B)$ is also σ-algebraic soft set over U.*

Proof Similar to proof of Theorem 3.5. □

Corollary 3.9 *Any number of intersection, union, $\wedge$ and $\vee$ of σ-algebraic soft sets is also σ-algebraic.*

Theorem 3.10 *Let (F, A) be a σ-algebraic soft set over U. Then its complement $(F, A)^c$ is also σ-algebraic soft set over U.*

Proof From Definitions 2.10 and 2.18 (b), we obtain that $(F, A)^c$ is a σ-algebraic soft set over U.

Theorem 3.11 *Let (F, A) and (G, B) be σ-algebraic soft sets over U. Then $(F, A) \times (G, B)$ is also σ-algebraic soft set over $U \times U$.*

Proof It is obvious from Definitions 2.12 and 2.23. □

Theorem 3.12 *Let (F, A) and (G, B) be soft sets over U. If (F, A) is similar to (G, B) and (F, A) is a σ-algebraic soft set, then (G, B) is a σ-algebraic soft set over U.*

Proof If $(F, A) \cong (G, B)$, then there exists a bijection $\phi : A \to B$ such that $F(e) = (G \circ \phi)(e)$ for every $e \in A$. Since (F, A) is a σ-algebraic soft set, then we obviously obtain that (G, B) is a σ-algebraic soft set over U.

Theorem 3.13 *Let $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ be measurable universes,, E_1 and E_2 be parameters sets, $\varphi : U_1 \to U_2$ measurable function and $\psi : E_1 \to E_2$ be a function. If (G, B) is a σ-algebraic soft set over U_2 then $(\varphi, \psi)^{-1}(G, B)$ is a σ-algebraic soft set over U_1.*

Proof Suppose that $\psi(e) \notin B$ for any $e \in E_1$, then we have $(\varphi^{-1}G)(e) = \varnothing$ from Definition 2.13 and $\varnothing \in \mathcal{A}_1$. Now, suppose that $\psi(e) \in B$. From Definition 2.13, we have $(\varphi^{-1}G)(e) = \varphi^{-1}[G(\psi(e))]$. Since (G, B) is σ-algebraic, i.e., $G(\psi(e)) \in \mathcal{A}_2$ for each $e \in E_1$ and φ is a measurable function, we obtain that $\varphi^{-1}[G(\psi(e))] \in \mathcal{A}_1$. Hence $(\varphi, \psi)^{-1}(G, B)$ is a σ-algebraic soft set over U_1.

Definition 3.14 Let U be a universe, $\mathfrak{B}(U)$ be a Borel algebra on U, (F, A) be a soft set over U. We call that (F, A) is a *Borelian soft set* if $F(e) \in \mathfrak{B}(U)$ for each $e \in A$.

In Shabir and Naz (2011), the concept of soft topology on a universe were defined by Shabir and Naz. They defined the soft topology as follows;

Definition 3.15 (Shabir and Naz 2011) Let $\mathcal{T}$ be the collection of soft sets over U, then $\mathcal{T}$ is said to be *soft topology* on U if

(1) $\widetilde{\Phi}$ and $\widetilde{U}$ belong to $\mathcal{T}$,
(2) the union of any number of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$,
(3) the intersection of any two soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The triplet $(U, \mathcal{T}, E)$ is called a *soft topological space* over U.

Shabir and Naz (2011) gave the following proposition.

Proposition 3.16 *(Shabir and Naz 2011) Let $(U, \mathcal{T}, E)$ be a soft topological space. Then the collection $\mathcal{T}_e = \{F(e) \mid (F, E) \in \mathcal{T}\}$ for each $e \in E$, defines a topology on U.*

We obtain following theorem from above proposition.

Theorem 3.17 *Let $(U, \mathcal{T}, E)$ be a soft topological space. Each element of $\mathcal{T}$ is a Borelian soft set.*

Proof From definition of Borel algebra and Proposition 3.16, it is obvious. □

Theorem 3.18 *Let (F, A) be a topological soft set over U. Then (F, A) is a Borelian soft set over U.*

Proof From Definition 2.16, if (F, A) is topological, then $\{F(e) \mid e \in A\}$ is topology on U. However, we have $\{F(e) \mid e \in A\} \subset \sigma(\{F(e) \mid e \in A\})$ where $\sigma(\{F(e) \mid e \in A\}) = \mathfrak{B}$ is a Borel algebra on U. Thus $F(e) \in \mathfrak{B}$ for all $e \in A$. Hence (F, A) is a Borelian soft set.

Theorem 3.19 *Let $(U, \mathcal{O})$ be a topological universe and (F, A) be an open soft set over U. Then (F, A) is a Borelian soft set over U.*

Proof Since $\mathcal{O} \subset \sigma(\mathcal{O})$, the result is obvious from Definitions 2.17 and 3.14. □

We can obtain relations among parameters using the measurement of sets via σ-algebraic soft set. One of them is an order relation. So we can sort among the parameters using the measure on the universe.

Definition 3.20 Let $(U, \mathcal{A}, \mu)$ be a measure space as a universe, E be a set of parameters, (F, A) be a σ-algebraic soft set. For each pair of parameters $e_1, e_2 \in A$, we called that e_1 is *less prefered* to e_2 which is denoted by $e_1 \preceq e_2$ if $\mu(F(e_1)) \le \mu(F(e_2))$.

The relation obtained in this way is a partial order relation (and so preference relation) on the parameter set $A \subseteq E$.

Example 3.21 Consider the universe $U = \{a, b, c, d, e, f\}$, the parameter set $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the σ-algebra $\mathcal{A} = \mathcal{P}(U)$ and the measure be cardinality of subsets of U. Let $(F, A) = \{1 = \{a, b\}, 2 = \{a, c, d\}, 4 = \{b, c, d, e\}, 7 = \{c\}\}$. Clearly, (F, A) is a σ-algebraic soft set over U. So, we obtain partial order relation on A via measure. Hence $7 \preceq 1 \preceq 2 \preceq 4$, i.e., 7 is less prefered to 1 and so.

Definition 3.22 Let $(U, \mathcal{A}, \mu)$ be a measure space as a universe, E be a set of parameters, (F, A) be a σ-algebraic soft set. For each pair of parameters $e_1, e_2 \in A$, we called that e_1 is *indiscernible* to e_2 which is denoted by $e_1 \sim e_2$ if $\mu(F(e_1)) = \mu(F(e_2))$. Otherwise they are *discernable*.

Example 3.23 Let $U = \{a, b, c, d\}$ be the initial universe, $E = \{1, 2, 3, 4, 5\}$ be the parameters set, $\mathcal{A} = \mathcal{P}(U)$ be the σ-algebra and $\mu(X) = \begin{cases} 1 \ , b \in X \\ 0 \ , b \notin X \end{cases}$ be the measure function on U for each $X \subseteq U$ and fixed $b \in U$. Now we define the σ-algebraic soft set over U as follows:

$$(F, E) = \{1 = \{a\}, 2 = \{c, d\}, 3 = \varnothing, 4 = \{b, c, d\}, 5 = \{a, d\}\}$$

So, we have $\mu(F(1)) = \mu(F(2)) = \mu(F(3)) = \mu(F(5)) = 0$ and $\mu(F(4)) = 1$. Thus we obtain the relationship among parameters as $1 \sim 2 \sim 3 \sim 5 \le 4$.

Note that, the indiscernibility relation $\sim$ is an equivalence relation on the parameter set A. The indiscernibility class of the parameter $e \in A$ is denoted by $[e]$. Hence, $A/\sim$ is a partition on A and we denote the partition $A/\sim$ as $[A]$. Therefore, if we have a σ-algebraic soft set over any measurable universe, then we can obtain a new σ-algebraic soft set using the partition A as follows.

Definition 3.24 Let $(U, \mathcal{A}, \mu)$ be a measurable universe, (F, A) be a σ-algebraic soft set over U where $A \subseteq E$. $(F^*, [A])$ is called *intersectional reduced soft set* of (F, A) such that $F^*([e]) = \bigcap_{e \sim e'} F(e')$.

Example 3.25 From Example 3.23, we obtain that reduced soft set of (F, A) is $(F^*, [A]) = \{[3] = \varnothing, [4] = \{b, c, d\}\}$.

Definition 3.26 Let $(U, \mathcal{A}, \mu)$ be a measurable universe, (F, A) be a σ-algebraic soft set over U where $A \subseteq E$. (F, A) is called *irreducible soft set* if $(F, A) \cong (F^*, [A])$.

Example 3.27 Let us construct our initial universe using the experiment of throwing two distinct dice. So, our initial universe as a sample space is $U = \{(x, y) \mid x, y \in \{1, 2, 3, 4, 5, 6\}\}$. Let's consider the parameter universe as sum of the number that appears on the dices, i.e., $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Thus, if we take the mapping $F : A \to \mathcal{P}(U)$ where $A = \{1, 2, 3, 4, 5, 6, 7\} \subset E$ such that

- $F(1) = \varnothing$,
- $F(2) = \{(1, 1)\}$,
- $F(3) = \{(1, 2), (2, 1)\}$,
- $F(4) = \{(1, 3), (3, 1), (2, 2)\}$,
- $F(5) = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$,
- $F(6) = \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$,
- $F(7) = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$,

then we obtain the soft set (F, A) over U. Now, let the measure μ on U be probability measure. At that case, we obtain $\mu(F(1)) = \frac{|F(1)|}{|U|} = \frac{0}{36} = 0$, $\mu(F(2)) = \frac{1}{36}$, $\mu(F(3)) = \frac{2}{36}$, $\mu(F(4)) = \frac{3}{36}$, $\mu(F(5)) = \frac{4}{36}$, $\mu(F(6)) = \frac{5}{36}$, $\mu(F(7)) = \frac{6}{36}$. Therefore, (F, A) is a σ-algebraic soft set over U and it is irreducible soft set over U with respect to probability measure on U.

Theorem 3.28 *Let $(U, \mathcal{A}, \mu)$ be a measurable universe, E be a parameter set and (F, A) be a σ-algebraic soft set over U. (F, A) is irreducible soft set if and only if the parameter set A is a chain in the E with respect to preference relation which is generated by measure on E.*

Proof Suppose that (F, A) is irreducible. Then $(F, A) \cong (F^*, [A])$. So, all parameters in A are self-equivalent, i.e., all parameters are discernable. Since (F, A) is σ-algebraic, then for all $e \in A$, $\mu(F(e)) \in \mathbb{R} \cup \{\infty\}$ and they can be compared with each other because the set of real number is totally ordered set. From Definition 3.20, we gain that all parameters in A can be compared with each other. Thus A is a totally ordered set. Hence A is a chain in E.

Conversely, suppose that A is a chain in E with respect to preference relation generated by the measure. At that case, all parameters in A are discernable, i.e., they are self-equivalent. Therefore, the natural projection $\pi : A \to [A]$ is a bijection. Hereby, the diagram

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{P}(U) \\ {[A]} \arrow[ur, "F^*"'] & \end{tikzcd}$$

is commutative, i.e., for all $e \in A$, $(F^* \circ \pi)(a) = F(a)$. Then $(F, A) \cong (F^*, [A])$. Hence (F, A) is irreducible. $\square$

Definition 3.29 Let $(U, \mathcal{A}, \mu)$ be a measure spaces as a universe, E be a set of parameters, $A \subseteq E$ and (F, A) be a σ-algebraic soft set over U. For any $e \in A$, we call that $\mu(F(e))$ is a *weight* of e in (F, A) and denoted by $w(e)$. Besides, sum of all weight of parameters in A is called *parametric weight* of (F, A) and denoted by $W(F, A)$.

Example 3.30 From, Example 3.21, we obtain $w(1) = |\{a, b\}| = 2$, $w(2) = 3$, $w(4) = 4$, $w(7) = 1$. So, parametric weight of (F, A), $W(F, A) = 2 + 3 + 4 + 1 = 10$.

Note that, the weight relation among parameters in A is an order relation as mentioned above, i.e., if $w(e_1) \leq w(e_2)$ then e_1 less preferred to e_2.

Definition 3.31 Let (F, A) be a σ-algebraic soft set over $(U, \mathcal{A}, \mu)$. For any $e \in A$, the ratio $\frac{w(e)}{W(F, A)}$ is called *impact* of the parameter e in (F, A) and denoted by $\imath(e)$.

Example 3.32 From Example 3.30, the impact of the parameter 1 in (F, A) is $\imath(1) = \frac{w(1)}{W(F, A)} = \frac{2}{10} = \frac{1}{5}$. And $\imath(2) = \frac{3}{10}$, $\imath(4) = \frac{4}{10}$, $\imath(7) = \frac{1}{10}$. Of course, impact of the parameter 4 is greater than the others and it is more preferred than others.

Theorem 3.33 $\imath(e_1) \leq \imath(e_2)$ *if and only if* $w(e_1) \leq w(e_2)$.

Proof It is obvious. $\square$

Theorem 3.34 *Let (F, A) and (G, B) be σ-algebraic soft sets over U. If $(F, A) \tilde{\subseteq} (G, B)$, then $W(F, A) \leq W(G, B)$.*

Proof If $(F, A)\widetilde{\subseteq}(G, B)$, then $A \subseteq B$ and $\forall e \in A$, $F(e) \subseteq G(e)$. Therefore, we obtain $\mu(F(e)) \leq \mu(G(e))$ for each $e \in A$. So,

$$\sum_{e\in A} \mu(F(e)) \leq \sum_{e\in A} \mu(G(e)).$$ □

Theorem 3.35 *Let (F, A) and (G, B) be σ-algebraic soft sets over $(U, \mathcal{A}, \mu)$. If $A \cap B = \varnothing$, then*

$$W\left((F, A)\widetilde{\cup}(G, B)\right) = W(F, A) + W(G, B).$$

Proof From Definition 2.7, we have $(F, A)\widetilde{\cup}(G, B) = (H, C)$ where $C = A \cup B$. Since $A \cap B = \varnothing$, we have $H(e) = F(e)$ for $e \in A - B = A$ and $H(e) = G(e)$ for $e \in B - A = B$. Thus we can easily see that $W(H, C) = W(F, A) + W(G, B)$. □

Theorem 3.36 *Let (F, A) and (G, B) be σ-algebraic soft sets over $(U, \mathcal{A}, \mu)$. If $(F, A) \cong (G, B)$ then $W(F, A) = W(G, B)$.*

Proof Since $(F, A) \cong (G, B)$, we have a bijection $\phi : A \to B$ such that $F = G\circ\phi$ from Definition 2.14. From Definition 3.29, we obtain that

$$\begin{aligned} W(F, A) &= \sum_{a\in A} \mu(F(a)) \\ &= \sum_{a\in A} \mu((G \circ \phi)(a)) \\ &= \sum_{b\in B} \mu(G(b)) \\ &= W(G, B). \end{aligned}$$ □

Theorem 3.37 *The parametric weight function is a (an outer) measure over the all σ-algebraic soft sets over U.*

Proof Define the mapping $W : \sigma\mathcal{S}(U; E) \to \mathbb{R} \cup \{\infty\}$ such that $W(F, A) = \sum_{a\in A} \mu(F(a))$. We should show the conditions of Definition 2.24. Then,

(1) if $\Phi_A \in \sigma\mathcal{S}(U; E)$ where $A \subseteq E$, then

$$W(\Phi_A) = \sum_{a\in A} \mu(\Phi(a)) = \sum_{a\in A} \mu(\varnothing) = 0.$$

(2) Suppose that $(F, A) \in \sigma\mathcal{S}(U; E)$, then $F(a) \in \mathcal{A}$ for all $a \in A$, and so $\mu(F(a)) \geq 0$ for all $a \in A$. Thus $W(F, A) = \sum_{a\in A} \mu(F(a)) \geq 0$.

(3) Let $\{(F_i, A_i)\}_{i\in I} \in \sigma\mathcal{S}(U; E)$ be a family of disjoint soft sets, i.e., $\bigcap_{i\in I} F_i(a_i) = \varnothing$ for all $i \in I$, $a_i \in A_i$.

Since $(U, \mathcal{A}, \mu)$ is a measure space, then we have

$$\begin{aligned} W\left(\bigcup_{i\in I}(F_i, A_i)\right) &= \sum_{a_i\in A_i} \mu\left(\bigcup_{i\in I} F_i(a_i)\right) \\ &= \sum_{a_i\in A_i} \left(\sum_{i\in I} \mu(F_i(a_i))\right) \\ &= \sum_{i\in I} \left(\sum_{a_i\in A_i} \mu(F_i(a_i))\right) \\ &= \sum_{i\in I} W(F_i, A_i) \end{aligned}$$

Hence, W is a measure over $\sigma\mathcal{S}(U; E)$. □

Suppose that, we have a soft set which is not σ-algebraic any universe. In the circumstances, we can produce a σ-algebraic soft set. For example,

Example 3.38 Consider the soft set

$$(F, E) = \{1 = \{a\}, 2 = \{c, d\}, 3 = \varnothing, 4 = \{b, c, d\}, 5 = \{a, d\}\}$$

over the set $U = \{a, b, c, d\}$ in Example 3.23. We say the set

$$F(E) = \{\{a\}, \{c, d\}, \varnothing, \{b, c, d\}, \{a, d\}\} \subset \mathcal{P}(U)$$

is value set of all parameter. We can generate a σ-algebra from $F(E)$ and the σ-algebra is $\sigma(F(E)) = \mathcal{P}(U)$. So, for all $e \in E$, $F(e) \in \sigma(F(E)) = \mathcal{P}(U)$. Hence we have achieved a σ-algebraic soft set from any soft set over U. If we take the number of elements of sets as a measure that we know that it is a measure on U, then we can order all the parameters in E. If we do this, we obtain that $3 \preceq 1 \preceq 2 \sim 5 \preceq 4$. Hereby, the intersectional reduced soft set of (F, E) is

$$(F^*, [E]) = \{[1]=\{a\}, [2] = \{d\}, [3] = \varnothing, [4]=\{b, c, d\}\}.$$

Consequently, among the parameters which is the most preferred is 4, and the least preferred parameter is 3. Besides, the parameters 2 and 5 are indiscernible.

Moreover, since $\mathcal{P}(U)$ is a σ-algebra on U, all soft sets over U is a σ-algebraic as we mentioned in Example 3.2. If we take the counting measure on $\mathcal{P}(U)$, we obtain the measurable space $(U, \mathcal{P}(U), \mu)$. In this manner, we can build a relationship among all interested parameters according to the counting measure on the initial universe. As in Example 3.38, if we take the soft set (F, E), then we obtain the relationship among all parameters in E as $3 \preceq 1 \preceq 2 \sim 5 \preceq 4$.

Corollary 3.39 *Let U be an initial universe, E be a parameter set. All interested parameters related with any soft set over U can be associated with each other with respect to counting measure.*

4 Conclusion

In this paper, we have defined the notion of σ-algebraic soft set and examined the set-theoretic operations among themselves over any given measurable universe. Using the measure on the initial universe, we have obtained relationships which are called *preference* and *indiscernibility relations* between parameters. So that the parameters have been characterized, i.e., we have pointed out which parameters are more preferable than others and which parameters indiscernible with each other. In a decision-making process, accurate ordering of parameters makes the decision-making easier and more comfortable. In addition to these, we have given a measurement which is called *parametric weight of a soft set* on all σ-algebraic soft sets and even soft sets over any given initial universe. The author hope that this article shed light on the scientist which is working in this area.

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References

Aktaş H, Çağman N (2007) Soft sets and soft groups. Inf Sci 177:2726–2735
Ali MI, Feng F, Liu XY, Min WK, Shabir M (2009) On some new operations in soft set theory. Comput Math Appl 57:1547–1553
Babitha KV, Sunil JJ (2010) Soft set relations and functions. Comput Math Appl 60:1840–1849
Chen D, Tsang ECC, Yeung DS, Wong X (2005) The parametrization reduction of soft sets and its applications. Comput Math Appl 49:757–763
Emelyanov E (2007) Introduction to measure theory and Lebesgue integration. Middle East Technical University Press, Ankara
Feng F, Jun YB, Zhao X (2008) Soft semirings. Comput Math Appl 56:2621–2628
Halmos PR (1950) Measure theory. D. Van Nostrand Company Inc., New York, p 314
Kharal A, Ahmad B (2009) Mappings on soft classes. New Math Nat Comput 7(3):471–481
Li Z, Chen H, Gao N (2013) The topological structure on soft sets. J Comput Anal Appl 15(4):746–752
Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45:555–562
Min WK (2012) Similarity in soft set theory. Appl Math Lett 25:310–314
Min WK (2014) Soft sets over a common topological universe. J Intell Fuzzy Syst 26:2099–2106
Molodtsov D (1999) Soft set theory-First results. Comput Math Appl 37:19–31
Pei D, Miao D (2005) From soft sets to information systems. In: Proceedings of IEEE international conference on granular computing 2:617–621
Shabir M, Naz M (2011) On soft topological spaces. Comput Math Appl 61(7):1786–1799
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
Zhu P, Wen Q (2010) Probabilistic soft sets. In: 2010 IEEE international conference on granular computing, pp 635–638