Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 66–74 (2017)
DOI: 10.1501/Commua1_0000000775
ISSN 1303–5991

A NOTE ON SOFT MODULES

CANAN CELEP YÜCEL AND UMMAHAN ACAR

Abstract. In this paper, essential soft submodule and complement of a soft submodule in a soft module are defined. The basic properties of such soft submodules are obtained. The notion of complement of soft submodules on soft modules is introduced. The relations between this and direct summand of soft modules are investigated.

1. Introduction

Dealing with uncertainties is a main problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with classical methods. Because, these classical methods have their inherent difficulties. To overcome these kinds of difficulties, Molodtsov [1] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Jun [2] and Park [3] pointed out several directions for the applications of soft sets. Moreover, many related concept with soft sets, especially soft set operations, have undergone tremendous studies. Maji et al. [4] studied some operations on the theory of soft sets. Ali et al. [5] introduced several new operations on soft sets. Sezgin and Atagün [6] and Ali et al. [7] studied on soft sets operations as well. Besides, Aktaş and Çağman [8] defined soft groups and obtained the main properties of these groups. Acar et al. [9] defined soft ring and applied the notion of soft sets by Molodtsov [1] to the ring theory. Sun et al. [10] defined the concept of soft modules and investigated their basic properties. This concept was also discussed by many authors. Atagün and Sezgin [11] introduced soft subrings and soft ideals of a ring by using Molodtsov's definition of the soft sets. Moreover, they introduced soft subfields of a field and soft submodule of a left R-module. Türkmen and Pancar [12] defined the notion of sum and direct sum of soft submodules, small soft submodules and radical of a soft module. Moreover, they showed that every finite sum of small soft submodules of a soft module is a

Received by the editors: Feb. 25, 2016, Accepted: June 26, 2016.
2010 *Mathematics Subject Classification.* 16D10; 03E75.
Key words and phrases. Soft module, Essential soft submodule, Complement soft submodule, Direct summand.

small soft module and proved that the class of small soft submodules of a soft module is closed under soft module homomorphisms. Furthermore, some interesting results in the theory of modules are still being explored currently. But the theory of essential and complement soft submodules have not yet been studied.

In this paper we first define essential soft submodule and complement soft submodule in a soft module. We then obtain the basic properties of these soft submodules. The existence of closure of a soft submodule is proved. In addition we prove that every direct summand of a soft module is a complement in the soft module and show that the converse is not true, in general.

2. Basic concepts of soft sets and soft modules

In [1] Molodtsov defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a subset of E.

Definition 2.1. [1] *A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow P(U)$.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [1].

At present, works on the soft set theory are prosing rapidly. The algebraic structure of this theory dealing with uncertainties has been studied by some authors ([4], [5]). We review some of the literature discussing the definitions of soft set, intersection, union and restricted intersection in the following.

Definition 2.2. [4] *For two soft sets (F, A) and (G, B) over a common universe U, we say that (G, B) is called a soft subset of (F, A), denoted by $(G, B)\widetilde{\subset}(F, A)$, if it satisfies the following:*

(i) *$B \subset A$.*
(ii) *$G(x) \subseteq F(x)$ for every $x \in B$.*

Definition 2.3. [4] *Let (F, A) and (G, B) be two soft sets over a common universe U. The intersection of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:*

(i) *$C = A \cap B$.*
(ii) *For all $x \in C$, $H(x) = F(x)$ or $G(x)$, (as both are same set).*

In this case, we write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Definition 2.4. [4] *Let (F, A) and (G, B) be two soft sets over a common universe U. The union of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:*

(i) *$C = A \cup B$.*

(ii) *For all* $x \in C$, $H(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B; \\ G(x) & \text{if } x \in B \setminus A; \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$

In this case, we write $(F, A) \,\widetilde{\cup}\, (G, B) = (H, C)$.

Definition 2.5. [5] *Let* (F, A) *and* (G, B) *be two soft sets over* M *such that* $A \cap B \neq \emptyset$. *The restricted intersection of* (F, A) *and* (G, B) *is denoted by* $(F, A) \Cap (G, B)$, *and is defined as*

$$(F, A) \Cap (G, B) = (H, A \cap B),$$

where $H(x) = F(x) \cap G(x)$ *for all* $x \in A \cap B$.

Now we will recall the definition of soft modules and their basic properties. Soft modules were defined by Sun et. al. [10]. This concept was also discussed by many authors (e.g. [6], [12]).

Throughout this paper, R is an associative ring with identity and all modules are unital left R-modules, unless otherwise stated. Let M be an R-module. By $N \leq M$, we mean that N is a submodule of M.

Definition 2.6. [10] *Let* (F, A) *be soft set over a module* M. (F, A) *is said to be a soft module over* M *if and only if* $F(x) \leq M$ *for all* $x \in A$.

Definition 2.7. [10] *Let* (F, A) *and* (G, B) *be two soft modules over* M. *Then* (G, B) *is soft submodule of* (F, A) *if*

(i) $B \subset A$.

(ii) $G(x) \leq F(x)$ *for all* $x \in B$.

This is denoted by $(G, B) \widetilde{\leq} (F, A)$.

If $(G, B) \widetilde{\leq} (F, A)$ *and* $(F, A) \widetilde{\leq} (G, B)$, *two soft sets* (F, A) *and* (G, B) *over* M *are called soft equal and is written* $(G, B) = (F, A)$.

Definition 2.8. [10] *Let* $(G, B) \widetilde{\leq} (F, A)$ *be soft modules over a module* M. (G, B) *is called maximal soft submodule of* (F, A) *if* $G(x)$ *is a maximal submodule of* $F(x)$ *for all* $x \in B$.

Definition 2.9. [11] *A soft module* (F, A) *over a module* M *is called whole (resp., trivial) if* $F(a) = M$ *(resp.,* $F(a) = \{0\}$*) for every* $a \in A$.

It is easy to show that every soft module has a trivial soft submodule.

Türkmen and Pancar [12] developed soft module theory, and introduced the sum and direct sum of soft submodules. They also defined the direct summand of soft modules. The following definitions and a theorem, which will be necessary for our next discussions, are taken from them.

Let M be an R-module and N_i $(i \in I)$ be a set of submodules N_i of M. The submodule, denoted by $< \cup_{i \in I} N_i >$, and defined by

$$\begin{aligned} < \cup_{i \in I} N_i > \; &= \; \{n_{i_1} + n_{i_2} + ... + n_{i_r} : n_{i_k} \in N_{i_k}, \\ &\quad \text{for some } r \in \mathbb{N}\}, \end{aligned}$$

is called the sum of the submodules which is denoted by $\sum_{i\in I} N_i$. Note that $<\emptyset>=\{0\}$.

Definition 2.10. [12] *Let (F,A) be a soft module over M and let $\{(F_i,A_i)\}_{i\in I}$ be any collection of soft submodules $(F_i,A_i)\widetilde{\leq}\ (F,A)$, where I is a nonempty set. The sum of the soft submodules (F_i,A_i) of (F,A) is defined as*

$$\sum_{i\in I}(F_i,A_i)=(H,\cup_{i\in I}A_i)$$

such that, for all $a\in\cup_{i\in I}A_i$, $H(a)=\sum_{i\in I(a)}F_i(a)$ where I(a) is the set of all elements $i\in I$ such that $a\in A_i$.

Definition 2.11. [12] *Let (F,A) be a soft module over a module M and $(G,B)\widetilde{\leq}(F,A)$. If there exists a soft submodule (T,C) of (F,A) such that $(G,B)\widetilde{+}(T,C)=(F,A)$ and $(G,B)\Cap(T,C)$ is trivial, then (G,B) is said to be a direct summand of (F,A) and denoted by $(G,B)\widetilde{\oplus}(T,C)=(F,A)$.*

Theorem 2.12. [12] *(The Soft Modular Law) Let (F,A), (G,B) and (T,C) be soft modules of a soft module over a module M with $(G,B)\widetilde{\leq}(F,A)$ and $A\cap C\neq\emptyset$. Then*

$$(F,A)\Cap[(T,C)\widetilde{+}(G,B)]=[(F,A)\Cap(T,C)]\widetilde{+}(G,B).$$

3. Essential and complement of soft submodules

In Dung et al. [13] essential submodules and complements are defined as follows, respectively:

Let M be a module. A submodule N of M is called essential in M, denoted by $N\leq_e M$, if $N\cap K\neq 0$ for every non-zero submodule K of M.

Let N be any submodule of M. A submodule H of M is called a complement of N (in M) if H is maximal in the collection of submodules Q of M such that $Q\cap N=0_M$ where 0_M is an identity element of (M. +).

In this section we extend the above definitions to the soft module theory. We first define essential soft submodules and complement of a submodule in a soft module. Then we develop various properties of such soft submodules.

Definition 3.1. *Let (F,A) be a soft module over M. A nontrivial soft submodule (G,B) of (F,A) is called essential soft submodule in (F,A), denoted by $(G,B)\widetilde{\leq}_e(F,A)$, if $(G,B)\Cap(T,C)$ is nontrivial for every nontrivial submodule (T,C) of (F,A) such that $B\cap C\neq\emptyset$.*

Clearly, if (F,A) is a nontrivial soft module over a module M, then (F,A) is an essential soft submodule of (F,A).

Example 3.2. *Consider left $\mathbb{Z}$- module $M={}_{\mathbb{Z}}\,\mathbb{Z}$. Let (F,A) be a soft set over the module M, where $A=\mathbb{Z}$ and $F:A\to P(M)$ is a set-valued function defined by $F(n)=M$ for all $n\in A$. Thus, (F,A) is a whole soft module over M. Let*

$B = 2\mathbb{Z}$ and $G : B \to P(M)$ is a function defined by $G(n) = 6\mathbb{Z}$. Then (G, B) is a soft submodule of (F, A). For every nontrivial submodule (T, C) of (F, A) such that $B \cap C \neq \emptyset$, $(G, B) \Cap (T, C)$ is nontrivial since $T(x) \cap G(x) \neq 0$ for all $x \in B \cap C$. Thus $(G, B)\widetilde{\leq}_e(F, A)$.

Lemma 3.3. *Let (F, A) be a soft module over M. If (K, B) and (L, C) are essential soft submodules of (F, A) such that $B \cap C \neq \emptyset$, then*

$$(K, B) \Cap (L, C)\widetilde{\leq}_e(F, A).$$

Proof. Let (N, E) be a nontrivial soft submodule of (F, A) such that $E \cap (B \cap C) \neq \emptyset$. Then, $E \cap B \neq \emptyset$. Since (K, B) is an essential soft submodule of (F, A), the soft submodule $(N, E) \Cap (K, B)$ is nontrivial. It follows from hypothesis that $[(N, E) \Cap (K, B)] \Cap (L, C)$ is nontrivial. Therefore, $(N, E) \Cap [(K, B) \Cap (L, C)] = [(N, E) \Cap (K, B)] \Cap (L, C)$. Hence, $(K, B) \Cap (L, C)$ is an essential soft submodule of (F, A). □

Corollary 3.4. *Let (F, A) be a soft module over M. If (K, B) is an essential soft submodule and (T, C) is nontrivial soft submodule of (F, A) such that $B \cap C \neq \emptyset$, then $K(x)$ is an essential submodule of $F(x)$ for every $x \in B \cap C$.*

Lemma 3.5. *Let (F, A) be a soft module over M and (G, B), (L, C), (X, E) and (T, S) are soft submodules of (F, A) with $B \cap E \neq \emptyset$ and $C \cap S \neq \emptyset$. If $(G, B)\widetilde{\leq}_e(L, C)$ and $(X, E)\widetilde{\leq}_e(T, S)$, then*

$$(G, B) \Cap (X, E)\widetilde{\leq}_e(L, C) \Cap (T, S).$$

Proof. Let (K, D) be a nontrivial soft submodule of $(L, C) \Cap (T, S)$ such that $D \cap B \cap E \neq \emptyset$. Then, $D \cap B \neq 0$. Hence $(K, D) \Cap (G, B)$ is nontrivial, since $(G, B)\widetilde{\leq}_e(L, C)$. Also $D \cap B \subseteq D \subseteq C \cap S \subseteq S$ and $(K, D) \Cap (G, B)\widetilde{\leq}(T, S)$. Since $(X, E)\widetilde{\leq}_e(T, S)$, the soft submodule $[(K, D) \Cap (G, B)] \Cap (X, E)$ is nontrivial. Hence $(G, B) \Cap (X, E)\widetilde{\leq}_e(L, C) \Cap (T, S)$. □

We now generalize the above result in the following corollary.

Corollary 3.6. *Let (F, A) be a soft module over M and (G_i, B_i) and (X_i, E_i) are soft submodules of (F, A) for every $1 \leq i \leq t$ with $B_i \cap E_i \neq \emptyset$. If every $(G_i, B_i)\widetilde{\leq}_e(X_i, E_i)$, then $(G_1, B_1) \Cap (G_2, B_2) \Cap ... \Cap (G_t, B_t)\widetilde{\leq}_e(X_1, E_1) \Cap (X_2, E_2) \Cap ... \Cap (X_t, E_t)$.*

Lemma 3.7. *Let (F, A) be a soft module over M and (L, C) be a nontrivial soft submodule of (F, A). If $(K, B)\widetilde{\leq}_e(F, A)$ and $B \cap C \neq \emptyset$, then $(K, B) \Cap (L, C)\widetilde{\leq}_e(L, C)$. In particular, (K, B) is a nontrivial soft submodule of (L, C), then $(K, B)\widetilde{\leq}_e(L, C)$.*

Proof. Let $(K, B)\widetilde{\leq}_e (F, A)$ and $(L, C)\widetilde{\leq}(F, A)$. For every nontrivial $(T, E)\widetilde{\leq}(F, A)$ with $B \cap E \neq \emptyset$, $(K, B) \Cap (T, E)$ is nontrivial. Therefore, for every $(X, I)\widetilde{\leq}(L, C)\widetilde{\leq}(F, A)$ with $I \cap B \neq \emptyset$, $(K, B) \Cap (X, I)$ is nontrivial. Hence $(X, I) \Cap ((K, B) \Cap (L, C)) =$

$((X, I) \Cap (K, B)) \Cap (L, C)$ is nontrivial, since $I \cap (B \cap C) \neq \emptyset$. Thus $(K, B) \Cap (L, C) \widetilde{\leq}_e (L, C)$. $\square$

In the next lemma we consider the transitive property of essential soft submodules.

Lemma 3.8. *Let (F, A) be a soft module over M. $(K, B) \widetilde{\leq}_e (F, A)$ if and only if*

$$(K, B) \widetilde{\leq}_e (N, E) \widetilde{\leq}_e (F, A).$$

Proof. Assume that $(K, B) \widetilde{\leq} (N, E) \widetilde{\leq} (F, A)$ and $(K, B) \widetilde{\leq}_e (F, A)$. It follows from Lemma 3.7 that $(K, B) \widetilde{\leq}_e (N, E)$. Now, suppose that (T, L) is nontrivial soft submodule of (F, A) such that $E \cap L \neq \emptyset$. Since $(K, B) \Cap (T, L)$ is nontrivial, the soft submodule $(N, E) \Cap (T, L)$ is nontrivial. Thus $(N, E) \widetilde{\leq}_e (F, A)$.

Conversely, let $(K, B) \widetilde{\leq}_e (N, E) \widetilde{\leq}_e (F, A)$, and (T, L) be a nontrivial soft submodule of (F, A) such that $L \cap B \cap E \neq \emptyset$. It can be seen that $(K, B) \Cap (T, L) = (K, B) \Cap ((N, E) \Cap (T, L))$. Since $(N, E) \widetilde{\leq}_e (F, A)$, the soft submodule $(K, B) \Cap ((N, E) \Cap (T, L))$ is nontrivial. Thus $(K, B) \Cap (T, L)$ is a nontrivial soft submodule of (F, A). Hence, $(K, B) \widetilde{\leq}_e (F, A)$. $\square$

Definition 3.9. *Let (F, A) be a soft module over M and $(G, B) \widetilde{\leq} (F, A)$. If $(T, C) \widetilde{\leq} (F, A)$ with $B \cap C \neq \emptyset$ is soft maximal w.r.t. $(G, B) \Cap (T, C)$ is trivial, then (T, C) is called complement of (G, B) in (F, A).*

The soft submodule (T, C) given above does not have to be unique. The following theorem states that every soft submodules of a soft module have a complement soft submodule.

Theorem 3.10. *Let (F, A) be a soft module over M. If (G, B) and (T, C) are soft submodules of (F, A) with $B \cap C \neq \emptyset$ such that $(G, B) \Cap (T, C)$ is trivial, then there exists a complement (K, D) of (G, B) in (F, A) such that $(T, C) \widetilde{\leq} (K, D)$.*

Proof. Let $E \cap B \neq \emptyset$ and $Z = \{(X, E) \widetilde{\leq} (F, A) : (T, C) \widetilde{\leq} (X, E)$, and $(X, E) \Cap (G, B)$ is trivial$\}$. So $Z \neq \emptyset$ since $(T, C) \in Z$. Clearly $(Z, \widetilde{\subseteq})$ is a complete ordered.

Now, assume that $\{(X_i, E_i) : i \in I\}$ is a chain in Z. Let $(S, V) = \cup_{i \in I} (X_i, E_i)$ such that $V = \cup_{i \in I} E_i$ and $V \cap B \neq \emptyset$. Then $(S, V) \widetilde{\leq} (F, A)$. So, $(T, C) \widetilde{\leq} (X_i, E_i)$ is trivial for every $i \in I$. Therefore, $\cup_{i \in I} (X_i, E_i) \Cap (G, B)$ is trivial and $(S, V) \in Z$. Hence (S, V) is an upper bound of $\{(X_i, E_i) : i \in I\}$ chain. So, by Zorn's Lemma, there exists a maximal element (K, D) of Z. It follows that (K, D) is a complement of (G, B) in (F, A) such that $(T, C) \widetilde{\leq} (K, D)$. $\square$

Definition 3.11. *Let (F, A) be a soft module over M. A soft submodule (N, E) of (F, A) is called a complement (in (F, A)) if there exists a soft submodule (T, C) of (F, A) such that (N, E) is a complement of (T, C) in (F, A), denoted by*

$$(N, E) \widetilde{\leq}_c (F, A).$$

Proposition 3.12. *Let (F, A) be a soft module over M. If (G, B) is a soft submodule of (F, A), then there exists a soft submodule (K, C) of (F, A), containing (G, B), such that $(G, B)\widetilde{\leq}_e (K, C)\widetilde{\leq}_c(F, A)$.*

Proof. Let (G', B') be a complement of (G, B) in (F, A). By Theorem 3.10, there exists a complement (K, C) of (G', B') in (F, A) with $(G, B) \subseteq (K, C)$. Let (L, T) be a nontrivial essential soft submodule of (K, C). Then $(G', B') \subseteq (L, T)+(G', B')$. Hence $((L, T) + (G', B')) \Cap (G, B)$ is nontrivial. Therefore $(G, B) \Cap ((L, T) + (G', B')) = (H,\ B \cap (T \cup B'))$ is nontrivial, where

$$H(a) = \begin{cases} G(a) \cap L(a) & \text{if } a \in B \cap (T \backslash B') \\ 0 & \text{if } a \in B \cap (B' \backslash T) \\ L(a) & \text{if } a \in B \cap (T \cap B') \ . \end{cases}$$

Then $(G, B) \Cap (L, T)$ is nontrivial since $0 \neq H(a) \in (G, B) \Cap (L, T)$. Thus $(G, B)\widetilde{\leq}_e (K, C)$. □

The soft submodule (K, C), the existence was proved in the above proposition, is called the *closure* of (G, B) in (F, A).

Proposition 3.13. *Let (F, A) be a soft module over M and (G, B) be a soft submodule of (F, A). $(G, B)\widetilde{\leq}_c(F, A)$ if and only if $(G, B) = (N, C)$ whenever (G, B) $\widetilde{\leq}_e(N, C)\widetilde{\leq}(F, A)$.*

Proof. Let $(G, B)\widetilde{\leq}_c(F, A)$ and $(G, B)\widetilde{\leq}_e(N, C)\widetilde{\leq}(F, A)$. Then there exists $(X, E)\widetilde{\leq}$ (F, A) where (G, B) is complement of (X, E) in (F, A). By Lemma 3.5, $(G, B) \Cap$ (X, E) $\widetilde{\leq}_e(N, C) \Cap (X, E)$. So $(N, C) \Cap (X, E)$ is trivial since $(G, B) \Cap (X, E)$ is trivial. Therefore (G, B) is maximal with respect to the property $(G, B) \Cap (X, E)$ is trivial. Hence $(G, B) = (N, C)$.

Conversely, let $(G, B)\widetilde{\leq}(F, A)$. By Proposition 3.13, there exists $(N, C)\widetilde{\leq}(F, A)$ such that $(G, B)\widetilde{\leq}_e(N, C)\widetilde{\leq}_c(F, A)$. Hence $(G, B)\widetilde{\leq}_c(F, A)$ since $(G, B) = (N, C)$. □

In the next proposition we consider the transitive property of complement soft submodules.

Proposition 3.14. *Let (F, A) be a soft module over M and (K, C), (N, E) be soft submodules of (F, A). If $(K, C)\widetilde{\leq}_c(N, E)$ and $(N, E)\widetilde{\leq}_c(F, A)$, then $(K, C)\widetilde{\leq}_c(F, A)$.*

Proof. Let (K, C) be a complement of (K', C') in (N, E) and (N, E) be a complement of (N', E') in (F, A). Suppose that $(K, C)\widetilde{\leq}_e(L, I)\widetilde{\leq}(F, A)$ such that $C \cap I \cap E' \neq \emptyset$. Therefore $(K, C) \Cap [(K', C') + (N', E')] = (H_1, C \cap (C' \cup E'))$ where

$$H_1(a) = \begin{cases} 0 & \text{if } a \in C \cap (C' \backslash E') \\ 0 & \text{if } a \in C \cap (E' \backslash C') \\ K(a) \cap (K'(a) + N'(a)) & \text{if } a \in C \cap (C' \cap E') \ . \end{cases}$$

Assume that $x \in K(a) \cap (K'(a) + N'(a))$. Then $x = k' + n'$ where $k' \in K'(a)$ and $n' \in N'(a)$. Therefore $x - k' = n' \in N(a) \cap N'(a) = 0$ and $x = k' \in K(a) \cap K'(a) = 0$ since $C \cap (C' \cap E') \subseteq C \cap C'$. Hence $(K, C) \Cap [(K', C') + (N', E')]$ is trivial. By Lemma 3.5, $(K, C) \Cap [(K', C') + (N', E')] \widetilde{\leq}_e (L, I) \Cap [(K', C') + (N', E')]$. Then $(L, I) \Cap [(K', C') + (N', E')]$ is trivial. By Theorem 2.12,

$$\begin{aligned}
&[(N, E) \Cap ((L, I) + (N', E'))] \Cap (K', C') = \\
&\quad [((N, E) \Cap (K', C'))] \Cap [(L, I) + (N', E')] = \\
&\quad (K', C') \Cap [(L, I) + (N', E')]. \qquad (1)
\end{aligned}$$

As above, $(K', C') \Cap [(L, I) + (N', E')]$ is trivial. Since (K, C) is a complement of (K', C') in (N, E), (K', C') is soft maximal w.r.t $(K, C) \Cap (K', C')$ is trivial. Since $(K, C) \subseteq (N, E) \Cap ((L, I) + (N', E'))$ and by (1) $(N, E) \Cap [(L, I) + (N', E')] = (K, C)$. Similarly, $(N', E') \Cap [(N, E) + (L, I)]$ is trivial. Hence $(N, E) + (L, I) = (N, E)$ since (N, E) is a complement of (N', E') in (F, A). So $(L, I) \widetilde{\leq} (N, E)$. Therefore $(L, I) = (L, I) \Cap [(L, I) + (N', E')] \widetilde{\leq} (N, E) \Cap [(L, I) + (N', E')] = (K, C)$. Thus $(K, C) \widetilde{\leq}_c (F, A)$ by Proposition 3.13. □

In light of the above information, we now prove that every direct summand is a complement in soft modules.

Theorem 3.15. *Let (F, A) be a soft module over M. Then every direct summand of (F, A) is a complement in (F, A).*

Proof. Let (G, B) be a direct summand of (F, A). Then there exist a soft submodule (T, C) of (F, A) such that $(G, B) \Cap (T, C)$ is trivial and $(G, B) \widetilde{\oplus} (T, C) = (F, A)$. Suppose $(K, D) \widetilde{\leq} (F, A)$ such that $(G, B) \subseteq (K, D) \subseteq (F, A)$ and $(K, D) \Cap (G, B)$ is trivial. Using Theorem 2.12, it can be shown that $(K, D) = (T, C)$. Hence (K, D) is a maximal soft submodule of (F, A). Thus (K, D) is complement of (G, B) in (F, A). □

In general, the converse of the above theorem is not true.

Example 3.16. *Let R be a ring, K be a field, and V be a vector space with dimension 2. Assume $R_R = \{\left[\begin{smallmatrix} k & v \\ 0 & k \end{smallmatrix}\right] : k \in K, v \in V = (v_1 K \oplus v_2 K)\}$ is a R-module, and $I = \{\left[\begin{smallmatrix} 0 & v_1 k \\ 0 & 0 \end{smallmatrix}\right] : k \in K\}$ and $J = \{\left[\begin{smallmatrix} 0 & v_2 k \\ 0 & 0 \end{smallmatrix}\right] : k \in K\}$ are submodules of R_R.*

We now define a soft set (F, A) over $U = R_R$ where $A = \{e_1, e_2\}$ such that $F(e_i) = R_R$, $i = 1, 2$. Given $B = \{e_1, e_2\} = A$, and a soft set (G, B) over U where

$$G(e_1) = G(e_2) = I,$$

and $C = \{e_2\}$, soft set (T, C) over U where

$$T(e_2) = J.$$

Then one can easily show that $(G, B) \widetilde{\leq} (F, A)$ and $(T, C) \widetilde{\leq} (F, A)$. Therefore $G(e_i) \cap T(e_2) = I \cap J = \{0\}$, for $i = 1, 2$. So $(G, B) \Cap (T, C)$ is trivial. Hence (G, B) is a complement of (T, C) in (F, A). Similarly (T, C) is a complement of (G, B) in

(F,A). But $(G,B)\widetilde{\oplus}(T,C) \neq (F,A)$. It follows that (G,B) is not a direct summand of (F,A).

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Current address: Canan CELEP YÜCEL: Department of Mathematics, Faculty of Arts and Sciences, Pamukkale University, 20070, Denizli, Turkey

E-mail address: `ccyucel@pau.edu.tr`

Current address: Ummahan ACAR: Department of Mathematics, Faculty of Arts and Sciences, Muğla Sıtkı Koçman University, 48170, Muğla, Turkey

E-mail address: `uacar@mu.edu.tr`