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Some topological properties of fuzzy cone metric spaces

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Abstract

We prove Baire's theorem for fuzzy cone metric spaces in the sense of Öner *et al.* [T. Öner, M. B. Kandemir, B. Tanay, J. Nonlinear Sci. Appl., **8** (2015), 610–616]. A necessary and sufficient condition for a fuzzy cone metric space to be precompact is given. We also show that every separable fuzzy cone metric space is second countable and that a subspace of a separable fuzzy cone metric space is separable.

Keywords: Fuzzy cone metric space, Baire's theorem, separable, second countable.
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1. Introduction

After Zadeh [13] introduced the theory of fuzzy sets, many authors have introduced and studied several notions of metric fuzziness ([2], [3], [4], [8], [9]) and metric cone fuzziness ([1], [10] from different points of view).

By modifying the concept of metric fuzziness introduced by George and Veeramani [4], Öner *et al.* [10] studied the notion of fuzzy cone metric spaces. In particular, they proved that every fuzzy cone metric space generates a Hausdorff first-countable topology.

Here we study further topological properties of these spaces whose fuzzy metric version can be found in [4], [5] and [6]. We show that every closed ball is a closed set and prove Baire's theorem for fuzzy cone metric spaces. Moreover, we prove that a fuzzy cone metric space is precompact if and only if every sequence in it has a Cauchy subsequence. Further, we show that $X_1 \times X_2$ is a complete fuzzy cone metric space if and only if X_1 and X_2 are complete fuzzy cone metric spaces. Finally it is proven that every separable fuzzy cone metric space is second countable and a subspace of a separable fuzzy cone metric space is separable.

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2. Preliminaries

Let E be a real Banach space, θ the zero of E and P a subset of E. Then P is called a cone [7] if and only if

1) P is closed, nonempty, and $P \neq \{\theta\}$;
2) if $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$;
3) if both $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone P, a partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if only if $y - x \in P$. The notation $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int\,(P)$ [7]. Throughout this paper, we assume that all the cones have nonempty interiors.

There are two kinds of cones: normal and nonnormal ones. A cone P is called normal if there exists a constant $K \geq 1$ such that for all $t, s \in E$, $\theta \preceq t \preceq s$ implies $\|t\| \leq K\|s\|$, and the least positive number K having this property is called normal constant of P [7]. It is clear that $K \geq 1$ [11].

According to [12], a binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies:

1) $*$ is associative and commutative;
2) $*$ is continuous;
3) $a * 1 = a$ for all $a \in [0, 1]$;
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

In [10], we generalized the concept of fuzzy metric space of George and Veeramani by replacing the $(0, \infty)$ interval by $int(P)$ where P is a cone as follows:

A fuzzy cone metric space is a 3-tuple $(X, M, *)$ such that P is a cone of E, X is nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times int\,(P)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s \in int\,(P)$ (that is $t \gg \theta$, $s \gg \theta$)

FCM1) $M\,(x, y, t) > 0$;
FCM2) $M\,(x, y, t) = 1$ if and only if $x = y$;
FCM3) $M\,(x, y, t) = M\,(y, x, t)$;
FCM4) $M\,(x, y, t) * M\,(y, z, s) \leq M\,(x, z, t + s)$;
FCM5) $M\,(x, y, .) : int\,(P) \longrightarrow [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy cone metric space, we will say that M is a fuzzy cone metric on X.

In [10], it was proven that every fuzzy cone metric space $(X, M, *)$ induces a Hausdorff first-countable topology τ_{fc} on X which has as a base the family of sets of the form $\{B(x, r, t) : x \in X, 0 < r < 1, t \gg \theta\}$, where $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for every r with $0 < r < 1$ and $t \gg \theta$.

A fuzzy cone metric space $(X, M, *)$ is called complete if every Cauchy sequence in it is convergent, where a sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $\varepsilon \in (0, 1)$ and any $t \gg \theta$ there exists a natural number n_0 such that $M\,(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$, and a sequence $\{x_n\}$ is said to converge to x if for any $t \gg \theta$ and any $r \in (0, 1)$ there exists a natural number n_0 such that $M\,(x_n, x, t) > 1 - r$ for all $n \geq n_0$ [10].

A sequence $\{x_n\}$ converges to x if and only if $\lim\limits_{n\to\infty} M\,(x_n, x, t) \to 1$ for each $t \gg \theta$ [10].

3. Results

Definition 3.1. Let $(X, M, *)$ be a fuzzy cone metric space. For $t \gg \theta$, the closed ball $B[x, r, t]$ with center x and radius $r \in (0, 1)$ is defined by $B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$.

Lemma 3.2. *Every closed ball in a fuzzy cone metric space $(X, M, *)$ is a closed set.*

Proof. Let $y \in \overline{B[x, r, t]}$. Since X is first countable, there exits a sequence $\{y_n\}$ in $B[x, r, t]$ converging to y. Therefore $M(y_n, y, t)$ converges to 1 for all $t \gg \theta$. For a given $\epsilon \gg 0$, we have

$$M(x, y, t + \epsilon) \geq M(x, y_n, t) * M(y_n, y, \epsilon).$$

Hence

$$M(x,y,t+\epsilon) \geq \lim_{n\to\infty} M(x,y_n,t) * \lim_{n\to\infty} M(y_n,y,\epsilon)$$
$$\geq (1-r) * 1 = 1-r.$$

(If $M(x,y_n,t)$ is bounded, then the sequence $\{y_n\}$ has a subsequence, which we again denote by $\{y_n\}$, for which $\lim_{n\to\infty} M(x,y_n,t)$ exists.) In particular for $n \in \mathbb{N}$, take $\epsilon = \frac{t}{n}$. Then

$$M(x,y,t+\frac{t}{n}) \geq (1-r)\,.$$

Hence

$$M(x,y,t) \geq \lim_{n\to\infty} M(x,y,t+\frac{t}{n}) \geq 1-r.$$

Thus $y \in B[x,r,t]$. Therefore $B[x,r,t]$ is a closed set. □

Theorem 3.3 (Baire's theorem)**.** *Let $(X,M,*)$ be a complete fuzzy cone metric space. Then the intersection of a countable number of dense open sets is dense.*

Proof. Let X be the given complete fuzzy cone metric space, B_0 a nonempty open set, and $D_1, D_2, D_3, \ldots$ dense open sets in X. Since D_1 is dense in X, we have $B_0 \cap D_1 \neq \emptyset$. Let $x_1 \in B_0 \cap D_1$. Since $B_0 \cap D_1$ is open, there exist $0 < r_1 < 1$, $t_1 \gg \theta$ such that $B(x_1, r_1, t_1) \subset B_0 \cap D_1$. Choose $r_1' < r_1$ and $t_1' = \min\{t_1, t_1/\|t_1\|\}$ such that $B[x_1, r_1', t_1'] \subset B_0 \cap D_1$. Let $B_1 = B(x_1, r_1', t_1')$. Since D_2 is dense in X, we have $B_1 \cap D_2 \neq \emptyset$. Let $x_2 \in B_1 \cap D_2$. Since $B_1 \cap D_2$ is open, there exist $0 < r_2 < 1/2$ and $t_2 \gg \theta$ such that $B(x_2, r_2, t_2) \subset B_1 \cap D_2$. Choose $r_2' < r_2$ and $t_2' = \min\{t_2, t_2/2\|t_2\|\}$ such that $B[x_2, r_2', t_2'] \subset B_1 \cap D_2$. Let $B_2 = B(x_2, r_2', t_2')$. Similarly proceeding by induction, we can find an $x_n \in B_{n-1} \cap D_n$. Since $B_{n-1} \cap D_n$ is open, there exist $0 < r_n < 1/n$, $t_n \gg \theta$ such that $B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n$. Choose an $r_n' < r_n$ and $t_n' = \min\{t_n, t_n/n\|t_n\|\}$ such that $B[x_n, r_n', t_n'] \subset B_{n-1} \cap D_n$. Let $B_n = B(x_n, r_n', t_n')$. Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t \gg \theta$, $0 < \varepsilon < 1$, choose an n_0 such that $t/n_0\|t\| \ll t, 1/n_0 < \varepsilon$. Then for $n \geq n_0$, $m \geq n$, we have

$$M(x_n, x_m, t) \geq M\left(x_n, x_m, \frac{t}{n_0\|t\|}\right) \geq 1 - \frac{1}{n} \geq 1 - \varepsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x$ in X. But $x_k \in B[x_n, r_n', t_n']$ for all $k \geq n$. Since $B[x_n, r_n', t_n']$ is closed, $x \in B[x_n, r_n', t_n'] \subset B_{n-1} \cap D_n$ for all n. Therefore $B_0 \cap (\bigcap_{n=1}^{\infty} D_n) \neq \emptyset$. Hence $\bigcap_{n=1}^{\infty} D_n$ is dense in X. □

Definition 3.4. A fuzzy cone metric space $(X,M,*)$ is called precompact if for each r, with $0 < r < 1$, and each $t \gg \theta$, there is a finite subset A of X, such that $X = \bigcup_{a\in A} B(a,r,t)$. In this case, we say that M is a precompact fuzzy cone metric on X.

Lemma 3.5. *A fuzzy cone metric space is precompact if and only if every sequence has a Cauchy subsequence.*

Proof. Suppose that $(X,M,*)$ is a precompact fuzzy cone metric space. Let $\{x_n\}$ be a sequence in X. For each $m \in \mathbb{N}$ there is a finite subset A_m of X such that $X = \bigcup_{a\in A_m} B(a, 1/m, t_0/m\|t_0\|)$ where $t_0 \gg \theta$ is a constant. Hence, for $m = 1$, there exists an $a_1 \in A_1$ and a subsequence $\{x_{1(n)}\}$ of $\{x_n\}$ such that $x_{1(n)} \in B(a_1, 1, t_0/\|t_0\|)$ for every $n \in \mathbb{N}$. Similarly, there exist an $a_2 \in A_2$ and a subsequence $\{x_{2(n)}\}$ of $\{x_{1(n)}\}$ such that $x_{2(n)} \in B(a_2, 1/2, t_0/2\|t_0\|)$ for every $n \in \mathbb{N}$. By continuing this process, we get that for $m \in \mathbb{N}$, $m > 1$, there is an $a_m \in A_m$ and a subsequence $\{x_{m(n)}\}$ of $\{x_{m-1(n)}\}$ such that $x_{m(n)} \in B(a_m, 1/m, t_0/m\|t_0\|)$ for every $n \in \mathbb{N}$. Now, consider the subsequence $\{x_{n(n)}\}$ of $\{x_n\}$. Given r with $0 < r < 1$ and $t \gg \theta$ there is an $n_0 \in \mathbb{N}$ such that $(1 - 1/n_0) * (1 - 1/n_0) > 1 - r$ and $2t_0/n_0\|t_0\| \ll t$. Then, for every $k, m \geq n_0$, we have

$$\begin{aligned}
M\left(x_{k(k)}, x_{m(m)}, t\right) &\geq M\left(x_{k(k)}, x_{m(m)}, \frac{2t_0}{n_0 \|t_0\|}\right) \\
&\geq M\left(x_{k(k)}, a_{n_0}, \frac{t_0}{n_0 \|t_0\|}\right) * M\left(a_{n_0}, x_{m(m)}, \frac{t_0}{n_0 \|t_0\|}\right) \\
&\geq \left(1 - \frac{1}{n_0}\right) * \left(1 - \frac{1}{n_0}\right) \\
&> 1 - r.
\end{aligned}$$

Hence $\left(x_{n(n)}\right)$ is a Cauchy sequence in $(X, M, *)$.

Conversely, suppose that $(X, M, *)$ is a nonprecompact fuzzy cone metric space. Then there exist an r with $0 < r < 1$ and $t \gg \theta$ such that for each finite subset A of X, we have $X \neq \bigcup_{a\in A} B(a, r, t)$. Fix $x_1 \in X$. There is an $x_2 \in X - B(x_1, r, t)$. Moreover, there is an $x_3 \in X - \bigcup_{k=1}^{2} B(x_k, r, t)$. By continuing this process, we construct a sequence $\{x_n\}$ of distinct points in X such that $x_{n+1} \notin \bigcup_{k=1}^{n} B(x_k, r, t)$ for every $n \in \mathbb{N}$. Therefore $\{x_n\}$ has no Cauchy subsequence. This completes the proof. □

Lemma 3.6. *Let $(X, M, *)$ be a fuzzy cone metric space. If a Cauchy sequence clusters around a point $x \in X$, then the sequence converges to x.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(X, M, *)$ having a cluster point $x \in X$. Then, there is a subsequence $\left\{x_{k(n)}\right\}$ of $\{x_n\}$ that converges to x with respect to τ_{fc}. Thus, given r with $0 < r < 1$ and $t \gg \theta$, there is an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $M\left(x, x_{k(n)}, t/2\right) > 1 - s$ where $s > 0$ satisfies $(1-s)*(1-s) > 1-r$. On the other hand, there is $n_1 \geq k(n_0)$ such that for each $n, m \geq n_1$, we have $M(x_n, x_m, t/2) > 1 - s$. Therefore, for each $n \geq n_1$, we have

$$\begin{aligned}
M(x, x_n, t) &\geq M(x, x_{k(n)}, \frac{t}{2}) * M(x_{k(n)}, x_n, \frac{t}{2}) \\
&\geq (1 - s) * (1 - s) \\
&> 1 - r.
\end{aligned}$$

We conclude that the Cauchy sequence $\{x_n\}$ converges to x. □

Proposition 3.7. *Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy cone metric spaces. For $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, let*

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

Then M is a fuzzy cone metric on $X_1 \times X_2$.

Proof. FCM1. Since $M_1(x_1, y_1, t) > 0$ and $M_2(x_2, y_2, t) > 0$, this implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0.$$

Therefore

$$M((x_1, x_2), (y_1, y_2), t) > 0.$$

FCM2. Suppose that for all $t \gg \theta$, $(x_1, y_1, t) = (x_2, y_2, t)$. This implies that $x_1 = y_1$ and $x_2 = y_2$ for all $t \gg \theta$. Hence

$$M_1(x_1, y_1, t) = 1$$

and

$$M_2(x_2, y_2, t) = 1.$$

It follows that

$$M((x_1, x_2), (y_1, y_2), t) = 1.$$

Conversely, suppose that $M((x_1, x_2), (y_1, y_2), t) = 1$. This implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1.$$

Since

$$0 < M_1(x_1, y_1, t) \leq 1$$

and

$$0 < M_2(x_2, y_2, t) \leq 1,$$

it follows that

$$M_1(x_1, y_1, t) = 1$$

and

$$M_2(x_2, y_2, t) = 1.$$

Thus $x_1 = y_1$ and $x_2 = y_2$. Therefore $(x_1, x_2) = (y_1, y_2)$.
FCM3. To prove that $M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t)$ we observe that

$$M_1(x_1, y_1, t) = M_1(y_1, x_1, t)$$

and

$$M_2(x_2, y_2, t) = M_2(y_2, x_2, t).$$

It follows that for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $t \gg \theta$

$$M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t).$$

FCM4. Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are fuzzy cone metric spaces, we have that

$$M_1(x_1, z_1, t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s)$$

and

$$M_2(x_2, z_2, t + s) \geq M_2(x_2, y_2, t) * M_2(y_2, z_2, s)$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$ and $t, s \gg \theta$. Therefore

$$\begin{aligned} M((x_1, x_2), (z_1, z_2), t + s) &= M_1(x_1, z_1, t + s) * M_2(x_2, z_2, t + s) \\ &\geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) * M_2(x_2, y_2, t) * M_2(y_2, z_2, s) \\ &\geq M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * M_1(y_1, z_1, s) * M_2(y_2, z_2, s) \\ &\geq M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, y_2), s). \end{aligned}$$

FCM5. Note that $M_1(x_1, y_1, t)$ and $M_2(x_2, y_2, t)$ are continuous with respect to t and $*$ is continuous too. It follows that

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$$

is also continuous. □

Proposition 3.8. *Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy cone metric spaces. We define*

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

Then M is a complete fuzzy cone metric on $X_1 \times X_2$ if and only if $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete.

Proof. Suppose that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy cone metric spaces. Let $\{a_n\}$ be a Cauchy sequence in $X_1 \times X_2$. Note that

$$a_n = (x_1^n, x_2^n)$$

and

$$a_m = (x_1^m, x_2^m).$$

Also, $M(a_n, a_m, t)$ converges to 1. This implies that

$$M((x_1^n, x_2^n), (x_1^m, x_2^m), t)$$

converges to 1 for each $t \gg \theta$. It follows that

$$M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$$

converges to 1 for each $t \gg \theta$. Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and also $M_2(x_2^n, x_2^m, t)$ converges to 1. Therefore $\{x_1^n\}$ is a Cauchy sequence in $(X_1, M_1, *)$ and $\{x_2^n\}$ is a Cauchy sequence in $(X_2, M_2, *)$. Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy cone metric spaces, there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1 for each $t \gg \theta$. Let $a = (x_1, x_2)$. Then $a \in X_1 \times X_2$. It follows that $M(a_n, a, t)$ converges to 1 for each $t \gg \theta$. This shows that $(X_1 \times X_2, M, *)$ is complete.

Conversely, suppose that $(X_1 \times X_2, M, *)$ is complete. We shall show that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete. Let $\{x_1^n\}$ and $\{x_2^n\}$ be Cauchy sequences in $(X_1, M_1, *)$ and $(X_2, M_2, *)$ respectively. Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and $M_2(x_2^n, x_2^m, t)$ converges to 1 for each $t \gg \theta$. It follows that

$$M((x_1^n, x_2^n), (x_1^m, x_2^m), t) = M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$$

converges to 1. Then (x_1^n, x_2^n) is a Cauchy sequence in $X_1 \times X_2$. Since $(X_1 \times X_2, M, *)$ is complete, there exists a pair $(x_1, x_2) \in X_1 \times X_2$ such that $M((x_1^n, x_2^n), (x_1, x_2), t)$ converges to 1. Clearly, $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_1^n, x_2, t)$ converges to 1. Hence $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete. This completes the proof. □

Theorem 3.9. *Every separable fuzzy cone metric space is second countable.*

Proof. Let $(X, M, *)$ be the given separable fuzzy cone metric space. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable dense subset of X. Consider

$$B = \left\{ B\left(a_j, \frac{1}{k}, \frac{t_1}{k\,\|t_1\|}\right) : j, k \in \mathbb{N} \right\}$$

where $t_1 \gg \theta$ is constant. Then B is countable. We claim that B is a base for the family of all open sets in X. Let G be an open set in X. Let $x \in G$; then there exists r with $0 < r < 1$ and $t \gg \theta$ such that $B(x, r, t) \subset G$. Since $r \in (0, 1)$, we can find an $s \in (0, 1)$ such that $(1 - s) * (1 - s) > (1 - r)$. Choose $m \in \mathbb{N}$ such that $1/m < s$ and $t_1/m\,\|t_1\| \ll \frac{t}{2}$. Since A is dense in X, there exists an $a_j \in A$ such that $a_j \in B(x, 1/m, t_1/m\,\|t_1\|)$. Now if $y \in B(a_j, 1/m, t_1/m\,\|t_1\|)$, then

$$\begin{aligned} M(x, y, t) &\geq M\left(x, a_j, \frac{t}{2}\right) * M\left(y, a_j, \frac{t}{2}\right) \\ &\geq M\left(x, a_j, \frac{t_1}{m\,\|t_1\|}\right) * M\left(y, a_j, \frac{t_1}{m\,\|t_1\|}\right) \\ &\geq \left(1 - \frac{1}{m}\right) * \left(1 - \frac{1}{m}\right) \\ &\geq (1 - s) * (1 - s) \\ &> (1 - r). \end{aligned}$$

Thus $y \in B(x, y, t)$ and hence B is a basis. □

Proposition 3.10. *A subspace of a separable fuzzy cone metric space is separable.*

Proof. Let X be a separable fuzzy cone metric space and Y a subspace of X. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. For arbitrary but fixed $n, k \in \mathbb{N}$, if there are points $x \in X$ such that $M(x_n, x, t_1/k\,\|t_1\|) > 1 - 1/k$, where $t_1 \gg \theta$ is constant, choose one of them and denote it by x_{nk}. Let $B = \{x_{nk} : n, k \in \mathbb{N}\}$; then B is countable. Now we claim that $Y \subset \overline{B}$. Let $y \in Y$. Given r with $0 < r < 1$ and $t \gg \theta$ we can find a $k \in \mathbb{N}$ such that $(1 - 1/k) * (1 - 1/k) > 1 - r$ and $t_1/k\,\|t_1\| \ll \frac{t}{2}$. Since A is dense in X, there exists an $m \in \mathbb{N}$ such that $M(x_m, y, t_1/k\|t_1\|) > 1 - 1/k$. But by definition of B, there exists an x_{mk} such that $M(x_{mk}, x_m, t_1/k\|t_1\|) > 1 - 1/k$. Now

$$\begin{aligned} M(x_{mk}, y, t) &\geq M\left(x_{mk}, x_m, \frac{t}{2}\right) * M\left(x_m, y, \frac{t}{2}\right) \\ &\geq M\left(x_{mk}, x_m, \frac{t_1}{k\|t_1\|}\right) * M\left(x_m, y, \frac{t_1}{k\|t_1\|}\right) \\ &\geq \left(1 - \frac{1}{k}\right) * \left(1 - \frac{1}{k}\right) \\ &> 1 - r. \end{aligned}$$

Thus $y \in \overline{B}$ and hence Y is separable. □

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