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Approximate solution of multi-pantograph equation with variable coefficients

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Abstract

This paper deals with the approximate solution of multi-pantograph equation with nonhomogenous term in terms of Taylor polynomials. The technique we have used is based on a Taylor matrix method. In addition, some numerical examples are presented to show the properties of the given method and the results are discussed.

MSC: 40A25; 65Q05; 41A10

Keywords: Taylor polynomials and series; Pantograph equations; Taylor matrix method; Delay differential equations

1. Introduction

Functional–differential equations with proportional delays are usually referred to as pantograph equations or generalized equations. The name *pantograph* originated from the work of Ockendon and Tayler [12] on the collection of current by the pantograph head of an electric locomotive.

These equations arise in many applications such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics and cell growth, among others. Properties of the analytic solution of these equations as well as numerical methods have been studied by several authors. For example, equations with variable coefficients are treated in [2,9,4].

In recent years, the Taylor method has been used to find the approximate solutions of differential, difference, integral and integro-differential-difference [5–7,11,13–15]. The basic motivation of this work is to apply the Taylor method to the nonhomogenous multi-pantograph equation with variable coefficients, which is extended of the multi-pantograph equation given in [8,1],

$$u'(t) = \lambda u(t) + \sum_{i=1}^{l} \mu_i(t)u(q_i t) + f(t), \quad t \geqslant 0 \tag{1}$$

under the condition $u(0) = \gamma$, where $\lambda, \gamma \in C$; $\mu_i(t)$ and $f(t)$ are analytical functions; $0 < q_i < 1$.

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In this study, our purpose is to find an approximate solution of the given problem in the series form

$$u(t)=\sum_{n=0}^{N}u_n t^n \tag{2}$$

so that the Taylor coefficients to be determined are

$$u_n=\frac{u_{(0)}^{(n)}}{n!},\quad n=0,1,2,\dots,N,\ N\in\mathbb{N}.$$

2. Fundamental matrix relations

Let us convert the expressions defined in (1) and (2) to the matrix forms. Now, let us assume that the functions $u(t)$ and $u'(t)$, respectively, can be expanded to Taylor series about $t=0$ in the forms

$$u(t)=\sum_{n=0}^{\infty}u_n t^n,\quad u_n=\frac{u_{(0)}^{(n)}}{n!} \tag{3}$$

and

$$u'(t)=\sum_{n=0}^{\infty}u'_n t^n, \tag{4}$$

where $u_n^{(0)}=u_n$.

First, let us derive the expression (3) with respect to t and then put $n\to n+1$.

$$u'(t)=\sum_{n=1}^{\infty}nu_n t^{n-1}=\sum_{n=0}^{\infty}(n+1)u_{n+1}t^n. \tag{5}$$

From (4) and (5), it is clear that

$$u'_n=(n+1)u_{n+1},\quad n=0,1,2,\dots\ . \tag{6}$$

which is the recurrence relation between the coefficients u'_n and $u_n^{(0)}$ of $u'_n(t)$ and $u(t)$, respectively. Now let us take $n=0,1,2,\dots,N$ and assume $u_n^{(k)}=0$, for $n>N$. Then system (6) can be transformed into the matrix form

$$\mathbf{U}^{(1)}=\mathbf{M}\mathbf{U}^{(0)}\equiv\mathbf{M}\mathbf{U}, \tag{7}$$

where

$$\mathbf{U}^{(1)}=\begin{bmatrix}u_0^{(1)}\\ u_1^{(1)}\\ \vdots\\ u_N^{(1)}\end{bmatrix},\quad \mathbf{M}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&2&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\cdots&N\\0&0&0&\cdots&0\end{bmatrix},\quad \mathbf{U}=\begin{bmatrix}u_0\\u_1\\ \vdots\\u_N\end{bmatrix}.$$

On the other hand, the solution expressed by (2) and its derivative (4) can be written in the matrix forms

$$[u(t)]=\mathbf{T}\mathbf{U}\quad\text{and}\quad[u'(t)]=\mathbf{T}\mathbf{U}^{(1)} \tag{8}$$

or using relation (7)

$$[u'(t)] = \mathbf{TMU}, \tag{9}$$

where

$$\mathbf{T} = [1 \quad t \quad t^2 \quad \cdots \quad t^N].$$

We can write the expressions $u(q_i t)$ and $\mu_i(t)$, respectively, as

$$u(q_i t) = \sum_{n=0}^{N} \frac{u^{(n)}(0)}{n!}(q_i)^n t^n = \sum_{n=0}^{N} u_n (q_i)^n t^n \tag{10}$$

and

$$\mu_i(t) = \sum_{k=0}^{N} a_{ik} t^k, \quad a_{ik} = \frac{\mu_i^{(k)}(0)}{k!}. \tag{11}$$

Hence from (10) and (11) we have

$$\mu_i(t) u(q_i t) = \sum_{k=0}^{N} \sum_{n=0}^{N} a_{ik}(q_i)^n t^{n+k} u_n, \quad t^{n+k} = 0 \text{ for } n+k > N$$

or the matrix form

$$[\mu_i(t) \;\; u(q_i t)] = \mathbf{TA_i U} \tag{12}$$

so that

$$\mathbf{T} = [1 \;\; t \;\; t^2 \;\; \cdots \;\; t^N],$$

$$\mathbf{U} = [u_0 \;\; u_1 \;\; u_2 \;\; \cdots \;\; u_N]^{\mathrm{T}},$$

$$\mathbf{A}_i = \begin{bmatrix} a_{i0}(q_i)^0 & 0 & 0 & \cdots & 0 \\ a_{i1}(q_i)^0 & a_{i0}(q_i)^1 & 0 & \cdots & 0 \\ a_{i2}(q_i)^0 & a_{i1}(q_i)^1 & a_{i0}(q_i)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,N}(q_i)^0 & a_{i,N-1}(q_i)^1 & a_{i,N-2}(q_i)^2 & \cdots & a_{i,0}(q_i)^N \end{bmatrix}.$$

Besides, we assume that the function $f(x)$ can be expanded as

$$f(x) = \sum_{n=0}^{N} f_n t^n, \quad f_n = \frac{f^{(n)}(0)}{n!}.$$

Then the matrix representation of $f(x)$becomes

$$[f(x)] = \mathbf{TF}, \tag{13}$$

where

$$\mathbf{F} = [f_0 \;\; f_1 \;\; \cdots \;\; f_N]^{\mathrm{T}}.$$

3. Method of solution

We now ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, we first substitute the matrix relations (8), (9), (12) and (13) into Eq. (1) and then simplify. Thus we have the fundamental matrix equation

$$\left\{\mathbf{M} - \lambda\mathbf{I} - \sum_{i=1}^{l} \mathbf{A_i}\right\}\mathbf{U} = \mathbf{F}, \tag{14}$$

where $\mathbf{I}$ is the identity matrix of order $N+1$; another matrices are defined in Sections 2.

The fundamental matrix (14) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $u_0, u_1, \ldots, u_N$. Briefly, we can write Eq. (14) in the form

$$\mathbf{WU} = \mathbf{F} \quad \text{or} \quad [\mathbf{W}; \mathbf{F}] \tag{15}$$

so that

$$\mathbf{W} = [w_{pq}] = \mathbf{M} - \lambda\mathbf{I} - \sum_{i=1}^{l} \mathbf{A}_i, \quad p, q = 0, 1, \ldots .$$

We can obtain the matrix form corresponding to the condition $u(0) = \gamma$ as, from the relation (8),

$$[1 \ \ 0 \ \ \cdots \ \ 0]\mathbf{U} = [\gamma] \tag{16}$$

or

$$[1 \ \ 0 \ \ \cdots \ \ 0 \ ; \ \ \gamma]$$

If the mixed condition

$$\sum_{r=0}^{R} c_r u(\gamma_r) = \gamma \tag{17}$$

is given, it can be written in the matrix form

$$\sum_{r=0}^{R} c_r \mathbf{T}(\gamma_r)\mathbf{U} = [\gamma], \tag{18}$$

where c_r, γ_r and γ are appropriate constants; the matrix $\mathbf{T}(\gamma_r)$ is

$$\mathbf{T}(\gamma_r) = [1 \ \ \gamma_r \ \ (\gamma_r)^2 \ \ \cdots \ \ (\gamma_r)^N].$$

Briefly, the matrix form for the condition (17) becomes

$$\mathbf{VU} = [\gamma] \quad \text{or} \quad [\mathbf{V}; \gamma], \tag{19}$$

where

$$\mathbf{V} = \sum_{r=0}^{R} c_r \mathbf{T}(\gamma_r) = [v_0 \ \ v_1 \ \ \cdots \ \ v_N].$$

To obtain the solution of Eq. (1) under the condition $u(0)=\gamma$ or the mixed condition (17), by replacing the row matrix (16) or (19) by the last row of matrix (15), we have the required augmented matrix [2,4]

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1 \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N} & ; & f_{N-1} \\ v_0 & v_1 & \cdots & v_N & ; & \gamma \end{bmatrix}, \tag{20}$$

where for condition $u(0)=\gamma$, $v_0=1$,$v_1=v_2=\cdots=v_N=0$.

If rank$\tilde{\mathbf{W}}=$ rank$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=N+1$ defined in (20), then we can write

$$U=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{F}}. \tag{21}$$

Thus the coefficients u_n, $n=0,1,2,\ldots,N$ are uniquely determined by Eq. (21). If $\det(\tilde{\mathbf{W}})=\mathbf{0}$, then there is no solution and the method cannot be used. Also, by means of systems we may obtain the particular solutions.

On the other hand, we can easily check the accuracy of the obtained solutions as follows [2]. Since the obtained polynomial expansion is an approximate solution, when the functions $u(t)$ and $u'(t)$are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $t=t_j$, $j=0,1,2,\ldots$.

$$E(t_j)=\left|u'(t_j)-\lambda u(t_j)-\sum_{i=1}^{l}\mu_i(t_j)u(q_it_j)-f(t_j)\right|\cong 0$$

or

$$E(t_j)\leqslant 10^{k_j}\quad (k_j \text{ isanypositiveinteger}).$$

If max $10^{k_j}=10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_j)$ at each of the points t_j becomes smaller than the prescribed 10^{-k}.

4. Illustrations

The method of this study is useful in finding the solutions of multi-pantograph equation with variable coefficients in terms of Taylor polynomials. We illustrate it by the following examples.

Example 1 (*Liu and Li [8]*). Let us first consider the equation

$$\begin{cases} u'(t)=-u(t)+\mu_1(t)u(0.5t)+\mu_2(t)u(0.25t) \\ u(0)=1 \end{cases}\qquad (0\leqslant t\leqslant 1), \tag{22}$$

where $\mu_1(t)=-\mathrm{e}^{-0.5t}\sin(0.5t)$, $\mu_2(t)=-2\mathrm{e}^{-0.75t}\cos(0.5t)\sin(0.25t)$ and approximate the solution $u(t)$ by the Taylor polynomial

$$u(t)=\sum_{n=0}^{5}u_nt^n,\quad u_n=\frac{u^{(n)}(0)}{n!}\ (n=0,1,\ldots,5),$$

where $\lambda=-1$, $q_1=0.5$, , $q_2=0.25$, $f(t)=0$. Then, the matrix form of Eq. (14) defined by

$$[\mathbf{M}-\lambda\mathbf{I}-\mathbf{A}_1-\mathbf{A}_2]\mathbf{U}=\mathbf{F},$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are matrices defined by

$$\mathbf{A}_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ -0.5 & 0 & 0 & 0 & 0 & 0\\ 0.25 & -0.25 & 0 & 0 & 0 & 0\\ -\dfrac{5}{12} & 0.125 & -0.125 & 0 & 0 & 0\\ 0 & -\dfrac{1}{48} & \dfrac{1}{16} & -0.0625 & 0 & 0\\ \dfrac{1}{960} & 0 & -\dfrac{1}{96} & \dfrac{1}{32} & -0.03125 & 0 \end{bmatrix},$$

$$\mathbf{A}_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ -0.5 & 0 & 0 & 0 & 0 & 0\\ 0.375 & -0.125 & 0 & 0 & 0 & 0\\ -\dfrac{7}{96} & 0.09375 & -0.03125 & 0 & 0 & 0\\ -0.015625 & -\dfrac{7}{384} & 0.0234375 & -0.0078125 & 0 & 0\\ \dfrac{161}{15\,360} & -0.00390625 & -\dfrac{7}{1536} & 0.005859375 & -0.001953125 & 0 \end{bmatrix}.$$

The augmented matrix forms of the conditions for $N=5$ is

$$[1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ ;\ 1].$$

Then, we obtain the augmented matrix (20) as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & ; & 0\\ 1 & 1 & 2 & 0 & 0 & 0 & ; & 0\\ -0.625 & 0.375 & 1 & 3 & 0 & 0 & ; & 0\\ 0.114583333 & -0.21875 & 0.15625 & 1 & 4 & 0 & ; & 0\\ 0.015625 & 0.0390625 & -0.0859375 & 0.0703125 & 1 & 5 & ; & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix}$$

Taking $N=5$, we obtain the approximate solution. The solution is

$$u(t)=1-t+0.3333333t^3-0.1666666t^4+0.0333333t^5.$$

Now let us find the solution of the problem (22) taking $N=5$. The values of this solution are compared with the results for $N=5$, $N=7$, $N=9$ given the exact solution $u(t)=\mathrm{e}^{-t}\cos t$ in Table 1.

Example 2 (*Evens and Raslan [3]*). Let us now consider the problem

$$u'(t)=\frac{1}{2}\mathrm{e}^{t/2}u\left(\frac{t}{2}\right)+\frac{1}{2}u(t),\quad u(0)=1\ \ (0\leqslant t\leqslant 1), \tag{23}$$

where $\lambda=\frac{1}{2}$, $q_1=\frac{1}{2}$, $f(t)=0$, $\mu_1(t)=\frac{1}{2}\mathrm{e}^{0.5t}$.

Table 1
Numeric results of Example 1

t_i	Present method			
	$N=5$	$E(t_i)$	$N=7$	$E(t_i)$
0	1	0	1	0
0.1	0.900316998	1.092×10^{-8}	0.900316999	3.6×10^{-10}
0.2	0.802410666	6.9082×10^{-7}	0.802410646	4.06×10^{-8}
0.3	0.707730999	7.754×10^{-6}	0.707730653	6.935×10^{-7}
0.4	0.617407999	4.2924×10^{-5}	0.617405399	5.1907×10^{-6}
0.5	0.532291666	1.6131×10^{-4}	0.532279266	2.4724×10^{-5}
0.6	0.452992000	4.7443×10^{-4}	0.452947565	8.8481×10^{-5}
0.7	0.379918999	1.1178×10^{-3}	0.379788279	2.5991×10^{-4}
0.8	0.313322666	2.5855×10^{-3}	0.312989785	6.6079×10^{-4}
0.9	0.253333333	5.1614×10^{-3}	0.252573798	1.5043×10^{-3}
1	0.199999999	9.5631×10^{-3}	0.198412698	3.1389×10^{-3}

t_i	Present method		Exact solution
	$N=9$	$E(t_i)$	$u(t)=\mathrm{e}^{-t}\cos t$
0	1	0	1
0.1	0.900316999	3×10^{-11}	0.900316999
0.2	0.802410647	1.3×10^{-9}	0.802410647
0.3	0.707730678	2.05×10^{-8}	0.707730678
0.4	0.617405654	1.434×10^{-7}	0.617405648
0.5	0.532280769	6.305×10^{-7}	0.532280730
0.6	0.452953943	2.058×10^{-6}	0.452953789
0.7	0.379909867	5.428×10^{-6}	0.379809389
0.8	0.313051744	1.212×10^{-5}	0.313050504
0.9	0.252730539	2.353×10^{-5}	0.252727753
1	0.198771632	4.003×10^{-5}	0.198766110

To find a Taylor polynomial solution of the problem above, we first take $N = 5$, and then proceed as before. Then we obtain the desired augmented matrix (20) as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & ; & 0 \\ -0.25 & -0.75 & 2 & 0 & 0 & 0 & ; & 0 \\ -0.0625 & -0.125 & -0.625 & 3 & 0 & 0 & ; & 0 \\ -0.010416666 & -0.03125 & -0.0625 & -0.5625 & 4 & 0 & ; & 0 \\ -0.013020833 & -0.00520333 & -0.015625 & -0.03125 & -0.53125 & 5 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix},$$

where $\tilde{\mathbf{W}}$ is the (6×6) matrix and $\tilde{\mathbf{F}}$ is the (6×1) column matrix. From the solutions of this system the coefficients $u_n (n = 0, 1, 2, \ldots, 5)$ are uniquely determined as

$$\mathbf{U} = [1\ \ 1\ \ 0.5\ \ 0.16666666\ \ 0.04166666\ \ 0.00833333]^{\mathrm{T}}.$$

By the substituting the obtained coefficients in (2) the solution of (23) becomes

$$u(t) = 1 + t + 0.5\,t^2 + 0.16666666t^3 + 0.04166666\,t^4 + 0.00833333t^5.$$

The comparison of the solutions (for $N = 5,\ 7,\ 9$) with exact solution $\exp(t)$ is given Table 2.

Table 2
Numeric results of Example 2

t_i	Present method			
	$N=5$	$E(t_i)$	$N=7$	$E(t_i)$
0	1	0	1	0
0.1	1.105170917	8.31×10^{-8}	1.105170918	4×10^{-10}
0.2	1.221402667	2.7119×10^{-6}	1.221402759	2.5×10^{-9}
0.3	1.349858775	2.0769×10^{-5}	1.349858806	4.41×10^{-8}
0.4	1.491818667	8.8292×10^{-5}	1.491824681	3.337×10^{-7}
0.5	1.648697917	2.7167×10^{-4}	1.648721168	1.6011×10^{-6}
0.6	1.822048000	6.8269×10^{-4}	1.822118354	5.7764×10^{-6}
0.7	2.013571417	1.4893×10^{-3}	2.013751158	1.711×10^{-5}
0.8	2.225130667	2.9313×10^{-3}	2.225536366	4.3879×10^{-5}
0.9	2.45875825	5.3335×10^{-3}	2.459591263	1.0079×10^{-4}
1	2.71666667	9.1216×10^{-3}	2.718253969	2.1226×10^{-4}

t_i	Present method		Exact solution
	$N=9$	$E(t_i)$	$u(t)=\mathrm{e}^t$
0	1	0	1
0.1	1.105170918	4×10^{-10}	1.105170918
0.2	1.221402759	5×10^{-10}	1.221402758
0.3	1.349858808	1×10^{-10}	1.349858808
0.4	1.491824698	1.2×10^{-9}	1.491824698
0.5	1.648721270	5.1×10^{-9}	1.648721271
0.6	1.822118799	2.93×10^{-8}	1.822118800
0.7	2.013752699	1.16×10^{-7}	2.013752707
0.8	2.225540897	3.87×10^{-7}	2.225540928
0.9	2.459603007	1.12×10^{-6}	2.459603111
1	2.718281527	2.91×10^{-6}	2.718281828

Example 3 (*Muroya et al. [10]*). Let us now consider the problem

$$u'(t) = -u(t) + \frac{q}{2}u(qt) - \frac{q}{2}\mathrm{e}^{-qt}, \quad u(0) = 1, \tag{24}$$

where $\lambda = -1$, $q_1 = q/2$, $f(t) = -(q/2)\mathrm{e}^{-qt}$, $\mu_1(t) = \frac{1}{2}\mathrm{e}^{0.5t}$.

Following the procedures in the previous examples, we get the approximate solution of problem (24) for $q = 0.9, 0.8, 0.5, 0.2$ and $N = 5$ as

$$u(t) = 1 - t + 0.5t^2 - 0.166666t^3 + 4.166666 \times 10^{-2}t^4 - 8.333333 \times 10^{-3}t^5.$$

Similarly, we have the approximate solution of problem (24) for $q = 0.9, 0.8, 0.5, 0.2$ and $N = 9$ as

$$\begin{aligned} u(t) = {} & 1 - t + 0.5\,t^2 - 0.166666t^3 + 4.166666 \times 10^{-2}t^4 - 8.333333 \times 10^{-3}t^5 + 1.388888 \times 10^{-3}t^6 \\ & - 1.98412698 \times 10^{-4}t^7 + 2.48015873 \times 10^{-5}t^8 - 2.755731922 \times 10^{-6}t^9. \end{aligned}$$

The comparison of the solutions given above with the exact solution $u(t) = \mathrm{e}^{-t}$ of the problem is given Table 3.

5. Error analysis

In this section, we will discuss the asymptotic behavior of error points $E(t_i)$ defined in Section 3 as the truncation limit N is increased. Fig. 1(a) shows the plot of the error points $E(t_i)$ for $N = 5$, 7 and 9. This plot clearly indicates that when we increase the truncation limit N, we have less error.

Table 3
Numeric results of Example 3

t_i	Exact solution $u(t)=e^{-t}$	Present method for $q=0.9, 0.8, 0.5, 0.2$			
		$N=5$	$E(t_i)$	$N=9$	$E(t_i)$
−1	2.718281828	2.716666667	9.141×10^{-3}	2.718281527	2.91×10^{-6}
−0.9	2.459603111	2.458758250	5.343×10^{-3}	2.459603007	1.12×10^{-6}
−0.8	2.225540928	2.225130667	2.935×10^{-3}	2.225540897	3.85×10^{-7}
−0.7	2.013752707	2.013571417	1.491×10^{-3}	2.013752699	1.15×10^{-7}
−0.6	1.822118800	1.822048000	6.834×10^{-4}	1.822118799	2.83×10^{-8}
−0.5	1.648721271	1.648697917	2.721×10^{-4}	1.648721271	5.9×10^{-9}
−0.4	1.491824698	1.491818667	8.834×10^{-5}	1.491824698	7×10^{-10}
−0.3	1.349858808	1.349857750	2.077×10^{-5}	1.349858808	1×10^{-10}
−0.2	1.221402758	1.221402667	2.712×10^{-6}	1.221402758	1×10^{-10}
−0.1	1.105170918	1.105170917	8.38×10^{-8}	1.105170918	0
0	1	1	0	1	0
0.1	0.904837418	0.904837416	8.27×10^{-8}	0.904837418	0
0.2	0.818730753	0.818730667	2.6235×10^{-6}	0.818730753	0
0.3	0.740818221	0.740817250	1.9746×10^{-5}	0.740818221	1×10^{-10}
0.4	0.670320046	0.670314667	8.2643×10^{-5}	0.670320046	1×10^{-10}
0.5	0.606530659	0.606510416	2.5029×10^{-4}	0.606530659	5.2×10^{-9}
0.6	0.548811636	0.548752000	6.1818×10^{-4}	0.548811635	2.69×10^{-8}
0.7	0.496585303	0.496436917	1.3263×10^{-3}	0.496585297	1.076×10^{-7}
0.8	0.449328964	0.449002667	2.5675×10^{-3}	0.449328937	3.559×10^{-7}
0.9	0.406569659	0.405916750	4.5942×10^{-3}	0.406569571	1.023×10^{-6}
1	0.367879441	0.366666667	7.7269×10^{-3}	0.367879189	2.629×10^{-6}

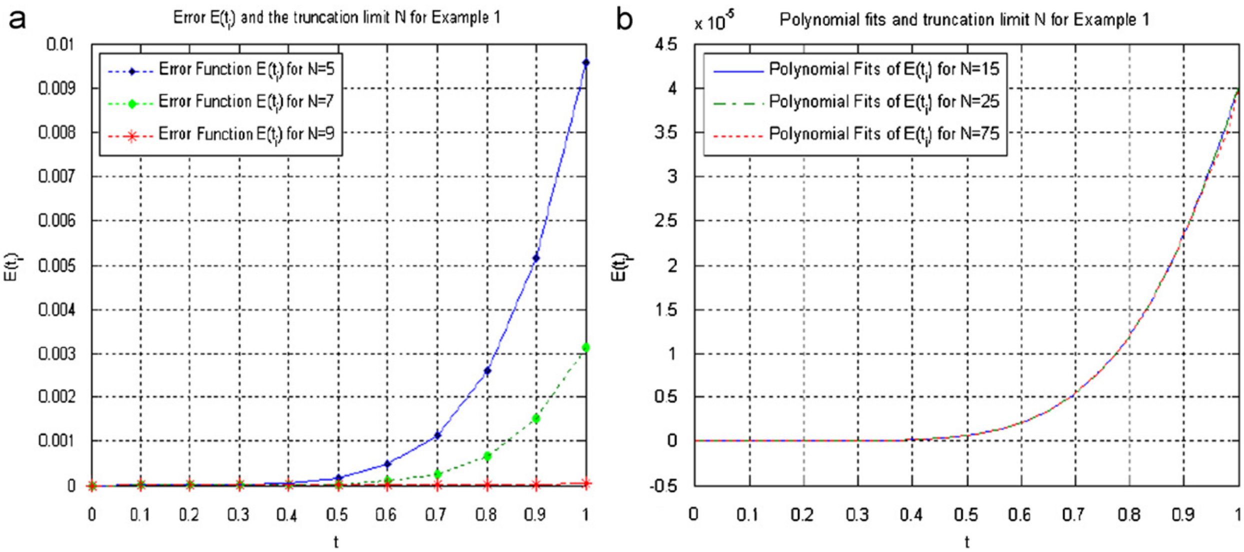

Fig. 1. (a) Error points $E(t_i)$ and the truncation limits for $N=5$, 7 and 9 in Example 1. (b) Increasing N does not effect the errors.

One question needs to be answered here is that how large we need to take N. To answer this question, we have used methods of curve fitting to estimate N and compare it to the error points $E(t_i)$. Since the approximate solutions $u(t)$of the given multi-pantograph equations are approximated polynomials depending on N, then the derivatives of these solutions $u'(t)$ are polynomials as well. Therefore, the first and second terms of $E(t_i)$ are polynomials. As a consequence of Eqs. (10) and (11), the functions $\mu_i(t_j)u(q_it_j)$ are polynomials. Since $f(t)$ is analytical function,

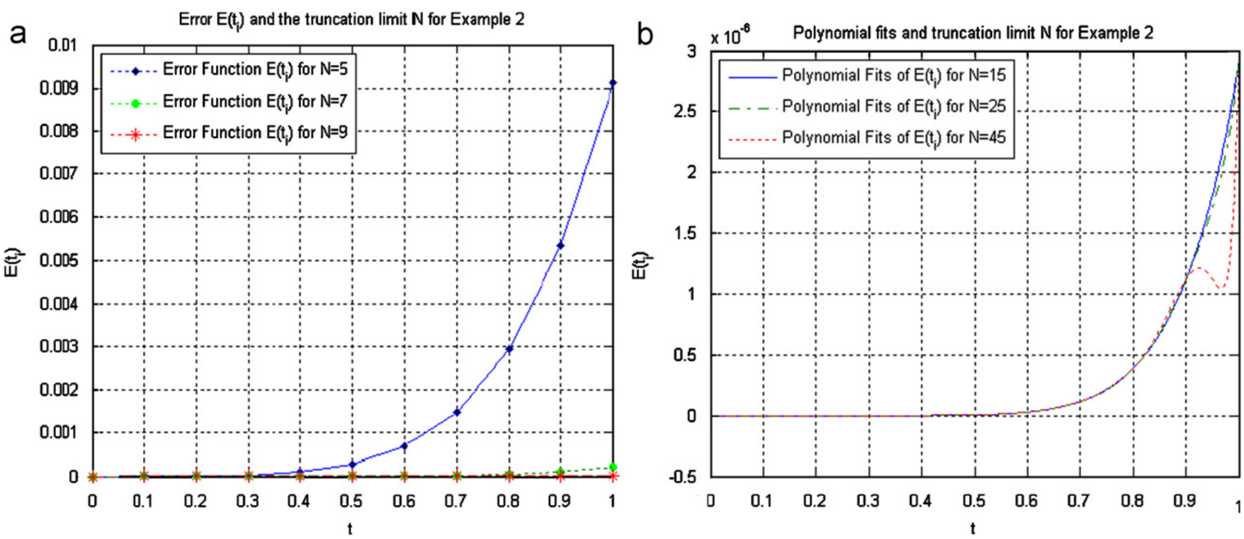

Fig. 2. (a) Error points $E(t_i)$ and the truncation limits for $N = 5$, 7 and 9 in Example 2. (b) Oscillation near the end point $t = 1$ can be expected for polynomials.

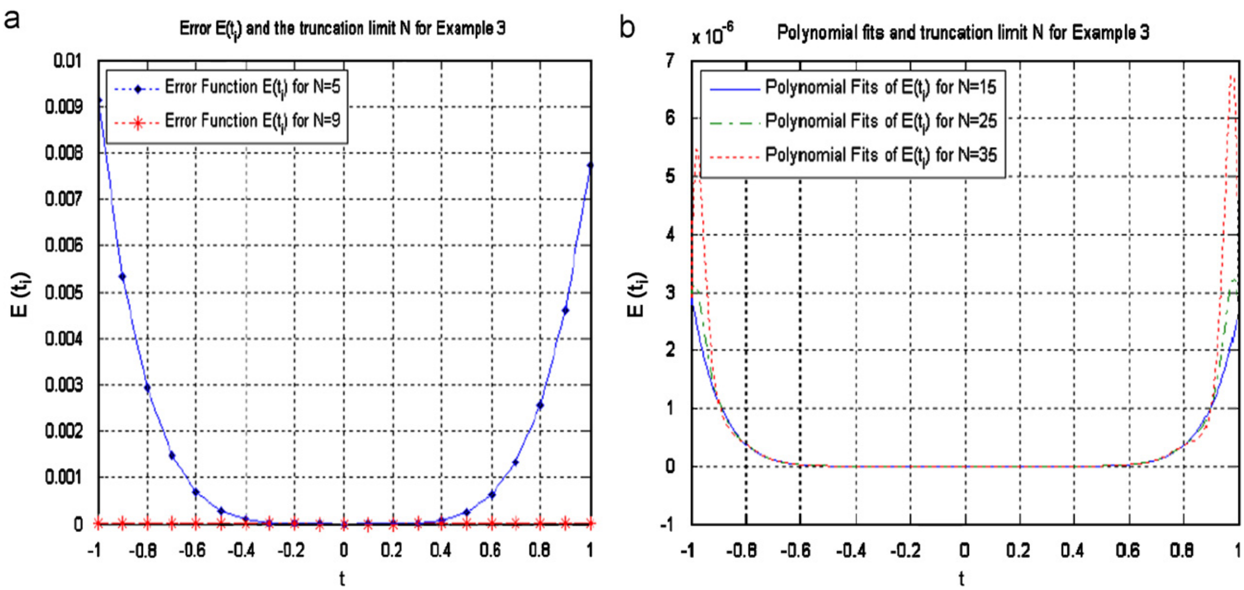

Fig. 3. (a) Error points $E(t_i)$ and the truncation limits for $N = 5$ and 9 in Example 3. (b) Increasing the truncation limit N does not effect the errors too much.

the remaining term in $E(t_i)$ can be curve fitted as polynomials. Using these ideas, we have found various degrees of polynomial fitting of error points $E(t_i)$ and we have compared the behaviors of error points and their polynomial fits as N increases in the interval $0 \leqslant t \leqslant 1$ as shown in Fig. 1(b). Fig. 1(b) shows that increasing N does not effect the errors very much.

Using similar ideas as in example 1 explained above, we can see from Fig. 2(b) that increasing N does not effect the errors very much. However, after $N = 45$, the polynomial fits show a tendency to oscillate to the end boundary point $t = 1$. This behavior can be expected in any polynomial numerical method.

Finally, we have similar results for Example 3. Fig. 3(a) shows the plot of the error points $E(t_i)$ for $N = 5$ and 9. Even for $N = 9$, we have very small errors. When we increase the truncation limit N, say $N = 25$, the polynomial starts oscillation near the end points as in Fig. 3(b).

6. Conclusions

Nonhomogenous multi-pantograph equation with variable coefficients are usually difficult to solve analytically. Then it is required to obtain the approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

The method presented in this study is a method for computing the coefficients in the Taylor expansion of the solution of a nonhomogenous multi-pantograph equation, and valid when the functions $\mu_i(t)$ and $f(t)$ are analytical functions.

The Taylor matrix method is an effective method for cases that the known functions have the Taylor series expansions at $x=0$. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial of degree N or less than N.

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