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Estimation in the Complementary Exponential Geometric Distribution Based on Progressive Type-II Censored Data

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Abstract
Complementary exponential geometric distribution has many applications in survival and reliability analysis. Due to its importance, in this study, we are aiming to estimate the parameters of this model based on progressive type-II censored observations. To do this, we applied the stochastic expectation maximization method and Newton–Raphson techniques for obtaining the maximum likelihood estimates. We also considered the estimation based on Bayesian method using several approximate: MCMC samples, Lindely approximation and Metropolis–Hasting algorithm. In addition, we considered the shrinkage estimators based on Bayesian and maximum likelihood estimators. Then, the HPD intervals for the parameters are constructed based on the posterior samples from the Metropolis–Hasting algorithm. In the sequel, we obtained the performance of different estimators in terms of biases, estimated risks and Pitman closeness via Monte Carlo simulation study. This paper will be ended up with a real data set example for illustration of our purpose.

Keywords Bayesian analysis · Complementary exponential geometric (CEG) distribution · Progressive type-II censoring · Maximum likelihood estimators · SEM algorithm · Shrinkage estimator

Mathematics Subject Classification 62N01 · 62N02

1 Introduction

Complementary risk (CR) problems arise naturally in a number of context, especially in problem of survival analysis, actuarial science, demography and industrial reliability

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[6]. In the classical complementary risk scenarios, the event of interest is related to causes which are not completely observed. Therefore, the lifetime of the event of interest is modeled as function of the available information, which is only the maximum ordered lifetime value among all causes. In the presence of CR in survival analysis, the risks are latent in the sense that there is no information about which factor was responsible for component failure, we observe only the maximum lifetime value among all risks. For example, when studying death on dialysis, receiving a kidney transplant is an event that competes with the event of interest such as heart failure, pulmonary embolism and stroke. In reliability, it observed only the maximum component lifetime of a parallel system, that is, the observable quantities for each component are the maximum lifetime value to failure among all risks and the cause of failure. For instance, in industrial applications, the failure of a device can be caused by several competing causes such as the failure of a component, contamination from dirt, an assembly error, harsh working environments, among others. For more literature on complementary risk problems, we refer the reader to Cox and Oakes [10], Crowder et al. [9], Goetghebeur and Ryan [14], Reiser et al. [34], Lawless [22] and Lu and Tsiatis [27, 28].

The complementary exponential geometric (CEG) model is derived as follows. Let M be a random variable denoting the number of failure causes, $m = 1, 2, \dots$, and considering M with geometrical distribution of probability given by

$$P(M = m) = \theta(1 - \theta)^{m-1}, \quad 0 < \theta < 1, \quad M = 1, 2, \dots .$$

Let us consider $x_i, \quad i = 1, 2, \dots$, realizations of random variable denoting the failure times, i.e., the time to event due to the ith complementary risk, with X_i has an exponential distribution with probability index λ, given by

$$f\left(x_i;\lambda\right) = \lambda \exp\left\{-\lambda x_i\right\}.$$

In the latent complementary risk scenario, the number of causes M and the lifetime x_i associated with a particular cause are not observable (latent variables), and only the maximum lifetime X among all causes is usually observed. So, it is only observed that the random variable is given by

$$X = \max\left\{X_i, \ 1 \le i \le M\right\}.$$

The CEG distribution, proposed recently by Louzada et al. [26] is useful model for modeling lifetime data. This distribution, with increasing failure rate, is complementary to the exponential geometric model given by Adamidis and Loukas [1]. Louzada et al. [26] showed that the probability distribution function of the two-parameter CEG random variables X is given by

$$f(x;\lambda,\theta) = \frac{\lambda\theta e^{-\lambda x}}{\left[e^{-\lambda x}(1-\theta)+\theta\right]^2}, \tag{1.1}$$

where $x > 0, \ \lambda > 0$ and $0 < \theta < 1$. Here λ and θ are the scale and shape parameters, respectively. It is denoted as $X \sim \text{CEG}(\lambda, \theta)$. The cumulative distribution function (CDF) and survival function of the $\text{CEG}(\lambda, \theta)$ are given by

$$F(x;\lambda,\theta) = 1 - \frac{e^{-\lambda x}}{\left[e^{-\lambda x}(1-\theta)+\theta\right]}, \tag{1.2}$$

$$S(x;\lambda,\theta)=\frac{e^{-\lambda x}}{\left[e^{-\lambda x}(1-\theta)+\theta\right]}, \tag{1.3}$$

respectively.

Where the lifetime associated with a particular risk is not observable, and it observed only the maximum lifetime value among all risks, then this distribution is used in latent complementary risks scenarios. Louzada et al. [26] discussed many properties of this model. But, they did not study about the estimation of the parameters based on censored data and prediction of future-order statistics. So, in this paper, we are aiming to cover these.

The rest of the paper is as follows: In Sect. 2, we discuss the maximum likelihood estimators of the parameters based on an expectation maximization (EM) and stochastic EM (SEM) algorithm. Section 3 deals with Bayes and shrinkage Bayes estimations assuming the Gamma and Beta priors. Prediction intervals for the survival time of future observation are also given in this section. Simulation studies as well as an illustrative example are the content of Sect. 4, and we gave our conclusion and the results in Sect. 5.

2 Maximum Likelihood Estimation

In this section, we determined the maximum likelihood estimates (MLEs) of the parameters of CEG distribution based on progressive type-II censored samples.

Suppose that n independent units are put on a test and that the lifetime distribution of each unit is given by $f\left(x_j;\lambda,\theta\right)$. Now consider the problem, the ordered m failures are observed under the progressively type-II censoring scheme plan $\boldsymbol{R}=\left(R_1,\dots,R_m\right)$, where each $R_j\geq 0$, $\sum_{j=1}^{m}R_j+m=n$. If the ordered m failures are denoted by $X_{1:m:n}<X_{2:m:n}<\dots<X_{m:m:n}$, then the likelihood function based on the observed sample $\boldsymbol{X}=\left(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n}\right)$ is given by

$$L(\boldsymbol{x};\lambda,\theta)=c\prod_{j=1}^{m}f\left(x_j;\lambda,\theta\right)\left[1-F\left(x_j;\lambda,\theta\right)\right]^{R_j}, \tag{2.1}$$

where $c=n\left(n-1-R_1\right)\left(n-2-R_1-R_2\right)\dots\left(n-\sum_{j=1}^{m-1}R_j-m+1\right)$. For simplicity, we denoted $x_{j:m:n}$ by x_j, $j=1,\dots,m$. Then, from Eqs. (1.1), (1.2) and (2.1), we can write the log-likelihood function of λ and θ based on progressive type-II censored observed sample $\boldsymbol{x}$ as:

$$\begin{aligned} l(\boldsymbol{x};\lambda,\theta)\propto\ & m\ln(\lambda)+m\ln(\theta)-\lambda\sum_{j=1}^{m}x_j \\ & -2\sum_{j=1}^{m}\ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]+\sum_{j=1}^{m}R_j\ln\left[\frac{e^{-\lambda x_j}}{e^{-\lambda x_j}(1-\theta)+\theta}\right]. \end{aligned} \tag{2.2}$$

MLEs of the parameters λ and θ can be obtained by solving two nonlinear equations simultaneously. In most cases, the estimators do not admit explicit. They have to be obtained by solving a two-dimensional optimization problem. It is observed that the standard Newton–Raphson (NR) algorithm has some problems such as does not converge in certain cases, a biased procedure, very sensitive to the initial values and also if the missing data are large then it is not convergent [31]. Little and Rubin [25] demonstrated that the estimation and maximization (EM) algorithm though converges slowly but is reasonably more reliable compared to the Newton–Raphson method, particularly when the missing data are relatively large. Here, we suggest using the EM algorithm to compute the desired MLEs.

2.1 EM and SEM Algorithm

The EM algorithm, originally proposed by Dempster et al. [12], is a very powerful tool in handling the incomplete data problem. The EM algorithm has two steps, E-step and M-step. For the E-step, one needs to compute the pseudo-log-likelihood function. It can be emerged from $\imath(\boldsymbol{w};\lambda,\theta)$ by substituting any function of z_{jk} say $g\left(z_{jk}\right)$ with $E\left[g\left(z_{jk}\right)|z_{jk}>x_j\right]$. And in the M-step, $E(\log \imath(\boldsymbol{w};\lambda,\theta))$ is maximized by taking the derivatives with respect to the parameters. McLachlan and Krishnan [30] gave a detailed discussion on EM algorithm and its applications.

We treat this problem as a missing value problem similarly as in Ng et al. [31]. The progressive type-II censoring can be viewed as an incomplete data set, and therefore, an EM algorithm is a good alternative to the NR method for numerically finding the MLEs. First, let us consider the observed and the censored data by $\boldsymbol{X}=\left(X_{1:m:n},\ldots,X_{m:m:n}\right)$ and $\boldsymbol{Z}=\left(Z_1,\ldots,Z_m\right)$, respectively, where each Z_j is $1\times R_j$ vector with $Z_j=\left(Z_{j1},\ldots,Z_{jR_j}\right)$ for $j=1,\ldots,m$, and they are not observable. The censored data vector $\boldsymbol{Z}$ can be thought of as missing data. The combination of $\boldsymbol{W}=(\boldsymbol{X},\boldsymbol{Z})$ forms the complete data set. The log-likelihood (LL) function based on the complete data is

$$\begin{aligned}\mathrm{LL}(\boldsymbol{w};\lambda,\theta) &\propto n\ln\lambda + n\ln\theta - \lambda\sum_{j=1}^{m}x_j - 2\sum_{j=1}^{m}\ln\left[\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right]\\ &- \lambda\sum_{j=1}^{m}\sum_{k=1}^{R_j}z_{jk}2\sum_{j=1}^{m}\sum_{k=1}^{R_j}\ln\left[\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta\right].\end{aligned} \tag{2.3}$$

The MLEs of the parameters λ and θ for complete sample $\boldsymbol{w}$ can be obtained by deriving the log-likelihood function in Eq. (2.3) with respect to λ and θ and equating the normal equations to 0 as follows:

$$\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{m} x_j + 2\sum_{j=1}^{m} \frac{x_j \mathrm{e}^{-\lambda x_j}(1-\theta)}{\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta} - \sum_{j=1}^{m}\sum_{k=1}^{R_j} z_{jk} + 2\sum_{j=1}^{m}\sum_{k=1}^{R_j} \frac{z_{jk}\mathrm{e}^{-\lambda z_{jk}}(1-\theta)}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta} = 0,$$

$$\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\theta} = \frac{n}{\theta} - 2\sum_{j=1}^{m} \frac{1-\mathrm{e}^{-\lambda x_j}}{\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta} - 2\sum_{j=1}^{m}\sum_{k=1}^{R_j} \frac{1-\mathrm{e}^{-\lambda z_{jk}}}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta} = 0.$$

In the E-step, the pseudo-log-likelihood function becomes,

$$\begin{aligned}\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\lambda} &= \frac{n}{\lambda} - \sum_{j=1}^{m} x_j + 2\sum_{j=1}^{m} \frac{x_j \mathrm{e}^{-\lambda x_j}(1-\theta)}{\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta} - \sum_{j=1}^{m}\sum_{k=1}^{R_j} E[z_{jk}|z_{jk} > x_j] \\ &\quad + 2\sum_{j=1}^{m}\sum_{k=1}^{R_j} E\left[\frac{z_{jk}\mathrm{e}^{-\lambda z_{jk}}(1-\theta)}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta}|z_{jk} > x_j\right] = 0,\end{aligned} \tag{2.4}$$

$$\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\theta} = \frac{n}{\theta} - 2\sum_{j=1}^{m} \frac{1-\mathrm{e}^{-\lambda x_j}}{\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta} - 2\sum_{j=1}^{m}\sum_{k=1}^{R_j} E\left[\frac{1-\mathrm{e}^{-\lambda z_{jk}}}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta}|z_{jk} > x_j\right] = 0. \tag{2.5}$$

We need the following result in sequel.

Theorem 2.1 *Given* $X_1 = x_1, \ldots, X_j = x_j$, *the conditional distribution of* Z_{jk}, $k = 1, \ldots, R_j$, *has form*

$$f_{Z|X}\left(z_j|X_1 = x_1, \ldots, X_j = x_j\right) = f_{Z|X}\left(z_j|X_j = x_j\right) = \frac{f(z_j|\lambda,\theta)}{[1-F(x_j|\lambda,\theta)]}, \tag{2.6}$$

where $z_j > x_j$ *and 0 otherwise.*

Proof The proof is straight forward. For details, see Ng et al. [31]. Using Theorem 2.1, we can write

$$E_1 = E\left[Z_{jk}|Z_{jk} > x_j\right] = \frac{\lambda\theta}{1-F(x_j|\lambda,\theta)} \times \int_{x_j}^{\infty} z_{jk}\mathrm{e}^{-\lambda z_j}\left[\mathrm{e}^{-\lambda z_j}(1-\theta)+\theta\right]^{-2}\mathrm{d}z_j. \tag{2.7}$$

And,

$$E_2 = E\left[\frac{z_{jk}\mathrm{e}^{-\lambda z_{jk}}(1-\theta)}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta}|Z_{jk} > x_j\right]$$

$$E_2 = \frac{\lambda\theta}{1 - F(x_j|\lambda,\theta)} \times \int_{x_j}^{\infty} \frac{z_{jk}\mathrm{e}^{-\lambda z_{jk}}(1-\theta)}{\left[\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta\right]} \frac{\mathrm{e}^{-\lambda z_j}}{\left[\mathrm{e}^{-\lambda z_j}(1-\theta)+\theta\right]^2}\mathrm{d}z_j. \tag{2.8}$$

And,

$$E_3 = E\left[\frac{1-\mathrm{e}^{-\lambda z_{jk}}}{\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta}|Z_{jk} > x_j\right] = \frac{\lambda\theta}{1-F(x_j|\lambda,\theta)} \times \int_{x_j}^{\infty} \frac{\left(1-\mathrm{e}^{-\lambda z_{jk}}\right)}{\left[\mathrm{e}^{-\lambda z_{jk}}(1-\theta)+\theta\right]} \times \frac{\mathrm{e}^{-\lambda z_j}}{\left[\mathrm{e}^{-\lambda z_j}(1-\theta)+\theta\right]^2}\mathrm{d}z_j. \tag{2.9}$$

Thus, in the M-step of the $(k+1)$th nonlinear iteration of the EM algorithm, the value of $\lambda^{(k+1)}$ is first obtained by solving the following equation:

$$\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\lambda} = \frac{n}{\lambda^{(k+1)}} - \sum_{j=1}^{m} x_j + 2\sum_{j=1}^{m} \frac{x_j\mathrm{e}^{-\lambda^{(k+1)}x_j}\left(1-\theta^{(k)}\right)}{\mathrm{e}^{-\lambda^{(k+1)}x_j}\left(1-\theta^{(k)}\right)+\theta^{(k)}} - \sum_{j=1}^{m} R_j E_1\left(x_j;\lambda^{(k)},\theta^{(k)}\right) + 2\sum_{j=1}^{m} R_j E_2\left(x_j;\lambda^{(k)},\theta^{(k)}\right) = 0.$$

Once $\lambda^{(k+1)}$ is obtained, then $\theta^{(k+1)}$ is obtained by solving the equation

$$\frac{\partial \mathrm{LL}(\boldsymbol{w};\lambda,\theta)}{\partial\theta} = \frac{n}{\theta^{(k+1)}} - 2\sum_{j=1}^{m} \frac{1-\mathrm{e}^{-\lambda^{(k+1)}x_j}}{\mathrm{e}^{-\lambda^{(k+1)}x_j}\left(1-\theta^{(k+1)}\right)+\theta^{(k+1)}} - 2\sum_{j=1}^{m} R_j E_3\left(x_j;\lambda^{(k+1)},\theta^{(k)}\right) = 0,$$

$\left(\lambda^{(k+1)},\theta^{(k+1)}\right)$ is then used as the new value of (λ,θ) in the subsequent iteration. Now the desired maximum likelihood estimates of λ and θ can be obtained using an iterative procedure which continues until $\left|\lambda^{(k+1)}-\lambda^{(k)}\right| + \left|\theta^{(k+1)}-\theta^{(k)}\right| < \varepsilon$, for some k, and a prespecified small value of ε.

A typical EM algorithm iteratively applies two steps; it is often having a simple closed form. However, in particular with high-dimensional data or increasing complexity for censored and lifetime models, one of the biggest disadvantages of EM algorithm is that it is only a local optimization procedure and can easily get stuck in a saddle point [40]. A possible solution to overcome the computational inefficiencies is to invoke stochastic EM algorithm suggested by Celeux and Diebolt [7], Nielsen [32] and Arabi Belaghi et al. [4]. It can be seen that the above EM expressions do not turn out to have closed form and therefore one needs to compute these expressions numerically. So, we used SEM algorithm to obtain maximum likelihood estimators.

The SEM algorithm is a two-step approach: the stochastic imputation step (S-step) and the maximization step (M-step). The main idea of the SEM algorithms is to replace the E-step by a stochastic step where the missing data **Z** are imputed with a single draw from the distribution of the missing data conditional on the observed **X**. The **Z** is then substituted to (2.3) to form the pseudo $\imath(\boldsymbol{w};\lambda,\theta)$ function, which is the optimized in the M-step to obtain $\left(\lambda^{k+1},\theta^{k+1}\right)$ for the next cycle. These two steps are repeated iteratively until a stationary distribution is reached for each parameter. The mean of this stationary distribution is considered as an estimator for the parameters. More formally, given the parameter estimate $\left(\lambda^{k},\theta^{k}\right)$ at the kth SEM cycle, $(k+1)$st cycle of the SEM algorithm evolves as follows:

S-Step Given the current $\left(\lambda^{k},\theta^{k}\right)$, simulate $\left(R_j\right)$ independent values from the conditional distribution $f_{Z|X}\left(x_{j:m:n};\lambda,\theta\right)$, respectively, for $j=1,\dots,m$ to form a realization of **Z**.

$$f_{Z|X}\left(x_{j:m:n};\lambda,\theta\right)=\frac{f\left(z_{jk};\lambda,\theta\right)}{1-F\left(x_{j:m:n};\lambda,\theta\right)}\ \text{ or }\ f_{Z|X}\left(x_{j:m:n};\lambda,\theta\right)=\frac{F\left(z_{jk};\lambda,\theta\right)-F\left(x_{j:m:n};\lambda,\theta\right)}{1-F\left(x_{j:m:n};\lambda,\theta\right)}$$

M-Step Maximize the pseudo $\imath(\boldsymbol{w};\lambda,\theta)$ function given (**X**, **Z**) to obtain $\left(\lambda^{k+1},\theta^{k+1}\right)$. □

2.2 Fisher Information Matrix

In this section, we present the observed Fisher information matrix obtained using the missing value principle of Louis [29]. The observed Fisher information matrix can be used to construct the asymptotic confidence intervals. The idea of missing information principle is as follows:

$$\text{Observed information} = \text{Complete information} - \text{Missing information}. \tag{2.10}$$

Let us use the following notation (regardless of denoting by bold notation): $\eta=(\lambda,\theta)$, X: the observed data, W: the complete data, $I_W(\eta)$: the complete information, $I_X(\eta)$: the observed information and $I_{W|X}(\eta)$: the missing information. Then, they can be expressed as follows:

$$I_X(\eta)=I_W(\eta)-I_{W|X}(\eta). \tag{2.11}$$

The complete information $I_W(\eta)$ is given by

$$I_W(\eta)=-E\left[\frac{\partial^2\imath(W;\eta)}{\partial\eta^2}\right].$$

The Fisher information matrix of the censored observations can be written as

$$I^{(j)}_{W|X}(\eta)=-E_{Z_j|X_j}\left[\frac{\partial^2\ln f_{Z_j}(z_j|X_j,\eta)}{\partial\eta^2}\right],$$

$$I_{W|X}(\eta)=\sum_{j=1}^{m}R_jI^{(j)}_{W|X}(\eta).$$

So we obtain the observed information as

$$I_X(\eta) = I_W(\eta) - I_{W|X}(\eta).$$

And naturally, the asymptotic variance covariance matrix of $\hat{\eta}$ can be obtained by inverting $I_X(\hat{\eta})$. The elements of matrices for $I_W(\eta)$ and $I_{W|X}(\eta)$ are denoted by $a_{ij}(\lambda, \theta)$ and $b_{ij}(\lambda, \theta)$.They are as follows:

$$a_{11} = \frac{n}{\lambda^2} + 2n\lambda\theta^2(1-\theta)\int_0^\infty \frac{x^2 e^{-2\lambda x}}{\left[e^{-\lambda x}(1-\theta)+\theta\right]^4}dx.$$

$$a_{22} = \frac{n}{\theta^2} - 2n\lambda\theta\int_0^\infty \frac{e^{-\lambda x}\left[1-e^{-\lambda x}\right]^2}{\left[e^{-\lambda x}(1-\theta)+\theta\right]^4}dx.$$

$$a_{12} = a_{21} = 2n\lambda\theta\int_0^\infty \frac{x e^{-2\lambda x}}{\left[e^{-\lambda x}(1-\theta)+\theta\right]^4}dx.$$

Now we provide $I_{W|X}(\eta)$. Since

$$I_{W|X}(\eta) = \sum_{j=1}^{m} R_j \begin{bmatrix} b_{11}\left(x_j;\lambda,\theta\right) & b_{12}\left(x_j;\lambda,\theta\right) \\ b_{21}\left(x_j;\lambda,\theta\right) & b_{22}\left(x_j;\lambda,\theta\right) \end{bmatrix}.$$

In which,

$$b_{11}\left(x_j;\lambda,\theta\right) = \frac{1}{\lambda^2} - \frac{x_j^2 e^{-\lambda x_j}\theta(1-\theta)}{\left[e^{-\lambda x_j}(1-\theta)+\theta\right]^2} + 2\lambda\theta^2(1-\theta)\int_0^\infty \frac{z_j^2 e^{-2\lambda z_j}}{\left[e^{-\lambda z_j}(1-\theta)+\theta\right]^4}dz_j.$$

$$b_{22}\left(x_j;\lambda,\theta\right) = \frac{1}{\theta^2} + \frac{\left[1-e^{-\lambda x_j}\right]^2}{\left[e^{-\lambda x_j}(1-\theta)+\theta\right]^2} - 2\lambda\theta\int_0^\infty \frac{e^{-\lambda z_j}\left[1-e^{-\lambda z_j}\right]^2}{\left[e^{-\lambda z_j}(1-\theta)+\theta\right]^4}dz_j.$$

$$b_{12}\left(x_j;\lambda,\theta\right) = b_{21}\left(x_j;\lambda,\theta\right) = -\frac{x_j e^{-\lambda x_j}}{\left[e^{-\lambda x_j}(1-\theta)+\theta\right]^2} + 2\lambda\theta\int_0^\infty \frac{z_j e^{-2\lambda z_j}}{\left[e^{-\lambda z_j}(1-\theta)+\theta\right]^4}dz_j.$$

Note that, we use the plug-in method of MLEs of λ and θ in finding the above values. Consequently the variance–covariance matrix of parameter η can be obtained by

$$I_X^{-1}(\eta) = \left[I_W(\eta) - I_{W|X}(\eta)\right]^{-1}. \tag{2.12}$$

Observe that we still need to obtain the integrations which may be cumbersome task. Next, we use the SEM algorithm to compute observed information matrix. We first generate the censored observations z_{ij} using Monte Carlo simulation from the conditional density as discussed in before. Subsequently the asymptotic variance–covariance matrix of the MLEs of the parameters can be obtained. Therefore, an approximate $(1-\alpha)100\%$ confidence interval for λ and θ is obtained as $\hat{\lambda} \pm z_{\alpha/2}\sqrt{\hat{V}\left(\hat{\lambda}\right)}$ and

$\hat{\theta} \pm z_{\alpha/2}\sqrt{\hat{V}(\hat{\theta})}$, where $z_{\alpha/2}$ is the $(\alpha/2)100$th percentile of standard normal distribution.

3 Bayes Estimates

In this section, we consider the Bayes estimates of the unknown parameters. For a Bayesian estimation of the parameters, one needs prior distributions for these parameters. These prior distributions depend upon the knowledge about the parameters and the experience of similar phenomena. When both the parameters of the model are unknown, a joint conjugate prior for the parameters does not exist. In view of the above, we propose to use independent gamma and beta priors for λ and θ, respectively. So, we assume the following independent priors:

$$\pi_1(\lambda) \propto \lambda^{a_1-1}e^{-b_1\lambda}, \quad \lambda > 0, \tag{3.1}$$

$$\pi_2(\theta) \propto \theta^{a_2-1}(1-\theta)^{b_2-1}, \quad 0 < \theta < 1. \tag{3.2}$$

Here, all the hyper-parameters a_1, b_1, a_2, b_2 are assumed to be known and non-negative. It can be observed that the non-informative priors of the parameters are the special case of the proposed prior distribution. Based on the observed sample $\{x_{1:m:n}, \cdots, x_{m:m:n}\}$, from the progressive type-II censoring scheme, the likelihood function becomes:

$$l(X;\lambda,\theta) \propto \lambda^m\theta^m e^{-\lambda\sum_{j=1}^m x_j} e^{-2\sum_{j=1}^m \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} e^{\sum_{j=1}^m R_j \ln\left[\frac{e^{-\lambda x_j}}{e^{-\lambda x_j}(1-\theta)+\theta}\right]}. \tag{3.3}$$

The joint posterior density functions of λ and θ can be written as

$$\begin{aligned}
\pi(\lambda,\theta|x) &\propto \lambda^{m+a_1-1}e^{-\lambda\left(\sum_{j=1}^m x_j+b_1\right)}\theta^{m+a_2-1}(1-\theta)^{b_2-1}e^{-2\sum_{j=1}^m \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} \\
&\quad \times e^{\sum_{j=1}^m R_j \ln\left[\frac{e^{-\lambda x_j}}{e^{-\lambda x_j}(1-\theta)+\theta}\right]} \\
&= \lambda^{m+a_1-1}e^{-\lambda\left(\sum_{j=1}^m x_j+b_1+\sum_{j=1}^m R_j x_j\right)}\theta^{m+a_2-1}(1-\theta)^{b_2-1}e^{-2\sum_{j=1}^m \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} \\
&\quad \times e^{-\sum_{j=1}^m R_j \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} \\
&= \text{gamma}\left(m+a_1, b_1+\sum_{j=1}^m x_j+\sum_{j=1}^m R_j x_j\right)\times \text{Beta}\left(m+a_2, b_2\right)\times h(\lambda,\theta),
\end{aligned} \tag{3.4}$$

where

$$h(\lambda,\theta) = e^{-2\sum_{j=1}^m \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} e^{-\sum_{j=1}^m R_j \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]}.$$

One may use the importance sampling method to obtain the MCMC samples and then compute the Bayes estimates. The simulation algorithm based on importance sampling is as follows.

- *Step 1* Generate λ from gamma $\sim \left(m + a_1, b_1 + \sum_{j=1}^{m} x_j + \sum_{j=1}^{m} R_j x_j\right)$.
- *Step 2* Generate θ from Beta $\sim \left(m + a_2, b_2\right)$.
- *Step 3* Compute $h(\lambda, \theta) = \mathrm{e}^{-2\sum_{j=1}^{m} \ln\left[e^{-\lambda x_j}(1-\theta)+\theta\right]} \mathrm{e}^{-\sum_{j=1}^{m} R_j \ln\left[\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right]}$.
- *Step 4* Do Steps 1 and 3 for N times.

The Bayes estimate of any function of λ and θ, say $g(\lambda, \theta)$, is evaluated as

$$E\left[g(\lambda, \theta)|x\right] = \frac{\sum g(\lambda, \theta) h(\lambda, \theta)}{\sum h(\lambda, \theta)}.$$

Therefore, the Bayes estimate of any function of λ and θ, say $g(\lambda, \theta)$, under the squared error loss function is:

$$L_1(\vartheta, \delta) = E\left[g(\lambda, \theta)|x\right] = \frac{\sum g\left(\lambda_j, \theta_j\right) h\left(\lambda_j, \theta_j\right)}{\sum h\left(\lambda_j, \theta_j\right)}.$$

One of the most commonly used asymmetric loss functions is the LINEX loss (LL) function, which is defined by:

$$L_2(\vartheta, \delta) = \exp\left(h(\delta - \vartheta)\right) - h(\delta - \vartheta) - 1, \quad h \neq 0.$$

The sign of parameter h represents the direction of asymmetry, and its magnitude reflects the degree of asymmetry. For $h < 0$, the underestimation is more serious than the overestimation, and for $h > 0$, the overestimation is more serious than the underestimation. For h close to zero, the LL function is approximately the SEL function. See Parsian and Kirmani [33].

In this case, the Bayes estimate of ϑ is obtained as:

$$\hat{\vartheta}_L = -\frac{1}{h} \ln\left[E_{\vartheta}\left(\mathrm{e}^{-h\vartheta}|X\right)\right],$$

provided the above exception exists.

Another commonly used asymmetric loss function is the general ENTROPY loss (EL) function given by:

$$L_3(\vartheta, \delta) = \left(\frac{\delta}{\vartheta}\right)^q - q \ln\left(\frac{\delta}{\vartheta}\right) - 1, \quad q \neq 0.$$

For $q > 0$, a positive error has a more serious effect than a negative error, and for $q < 0$, a negative error has a more serious effect than a positive error. Note that for $q = -1$, the Bayes estimate coincides with the Bayes estimate under the SEL function. In this case, the Bayes estimate of ϑ is obtained as:

$$\hat{\vartheta}_E = \left[E_{\vartheta}(\vartheta^{-q}|X)\right]^{-\frac{1}{q}}$$

provided the above exception exists.

3.1 Shrinkage Preliminary Test Estimator

In problems of statistical inference, there may exist some known prior information on some (all) of the parameters, which are usually incorporated in the model as a constraint, giving rise to restricted models. The estimators resulting from restricted (unrestricted) model are known as the restricted (unrestricted) estimators. Mostly the validity of a restricted estimator is under suspicion, resulting to make a preliminary test on the restrictions. Bancroft [5] pioneered the use of the preliminary test estimator (PTE) to eliminate such doubt, and further developments appeared in the works of Saleh and Sen [37], Saleh and Kibria [36], Kibria [15], Kibria and Saleh [16–20] and Arabi Belaghi et al. [2, 3].

Here, we suppose there exists some non-sample prior information with form of $\lambda = \lambda_0$ and we are interested in estimating λ using such information. So, we can run the following simple hypotheses to check the accuracy of this information:

$$\begin{cases} H_0 \, : \, \lambda = \lambda_0, \\ H_1 \, : \, \lambda \neq \lambda_0. \end{cases}$$

It is demonstrated that constructing shrinkage estimators for λ based on fixed alternatives $H_1 \, : \, \lambda = \lambda_0 + \delta$, for a fixed δ, does not offer substantial performance change compared to $\hat{\lambda}$. In other words, the asymptotic distribution of shrinkage estimator coincides with that of $\hat{\lambda}$ (see Saleh [35] for more details). To overcome this problem, we consider local alternatives with form

$$A_{(r)} \, : \, \lambda_{(r)} = \lambda_0 + r^{-\frac{1}{2}} \delta,$$

where δ is a fixed number.

Under H_0, $\sqrt{r}\left(\hat{\lambda} - \lambda\right)$ is asymptotically $N\left(0, \sigma^2\left(\hat{\lambda}\right)\right)$ and the test statistics can be defined as

$$W_r = \left(\frac{\sqrt{r}\left(\hat{\lambda} - \lambda\right)}{\sigma\left(\hat{\lambda}\right)} \right)^2,$$

where $\hat{\lambda}$ is MLE of λ resulted from SEM method and $\sigma^2\left(\hat{\lambda}\right)$ is the associated variance of $\hat{\lambda}$ that is obtained from the missing information principle. Based on the asymptotic distribution of W_r, we reject H_0 when $W_r > \chi_1^2(\gamma)$, where γ is the type-one error that prespecified by the researchers and $\chi_1^2(\gamma)$ is the γ the upper quantile of chi-square distribution with one degree of freedom.

The asymptotic distribution of W_r converges to a non-central chi-square distribution with one degree of freedom and non-centrality parameter $\Delta^2/2$, where

$$\Delta^2 = \frac{\delta^2}{\sigma^2\left(\hat{\lambda}\right)}.$$

Note that $\sigma^2\left(\hat{\lambda}\right)$ is obtained from (2.11). Thus, we define the shrinkage preliminary test estimator (PTE) of λ as

$$\hat{\lambda}^{\text{EM.SPT}} = \omega\lambda_0 + (1-\omega)\hat{\lambda}_{\text{EM}} I\left(W_r < \chi_1^2(\gamma)\right), \tag{3.5}$$

and

$$\hat{\lambda}^{\text{B.SPT}} = \omega\lambda_0 + (1-\omega)\hat{\lambda}_{\text{Bayes}} I\left(W_r < \chi_1^2(\gamma)\right). \tag{3.6}$$

In which $\omega \in [0,1]$ and $\hat{\lambda}_{\text{Bayes}}$ is the Bayes estimate of λ (see Arabi Belaghi et al. [2, 3], for more details about the construction of PTEs). We call the $\hat{\lambda}^{\text{B.SPT}}$ as the Bayesian shrinkage preliminary test estimators (BSPTE). The shrinkage PTE (SPTE) of θ is also defined in a similar fashion as in (3.5), which is not given here. Shrinkage and preliminary test estimators are extensively studied by Saleh [35] and Saleh et al. [38].

3.2 Lindley Approximation Method

In previous section, we obtained various Bayesian estimates of λ and θ based on progressive type-II censored observations. We notice that these estimates are in the form of ratio of two integrals. In practice, by applying Lindley method (see Lindley [24]) one can approximate all these Bayesian estimates. For the sake of completeness, we briefly discuss the method below and then apply it to evaluate corresponding approximate Bayesian estimates. Since the Bayesian estimates are in the form of ratio of two integrals, we consider the function $I(\boldsymbol{X})$ defined as

$$I(\boldsymbol{X}) = \frac{\int_0^\infty \int_0^\infty u(\lambda,\theta) \mathrm{e}^{l(\lambda,\theta|X)+\rho(\lambda,\theta)} \mathrm{d}\lambda \mathrm{d}\theta}{\int_0^\infty \int_0^\infty \mathrm{e}^{l(\lambda,\theta|X)+\rho(\lambda,\theta)} \mathrm{d}\lambda \mathrm{d}\theta},$$

where $u(\lambda,\theta)$ is function of λ and θ only and $l(\lambda,\theta|X)$ is the log-likelihood (defined by Eq. 2.2) and $\rho(\lambda,\theta) = \log\pi(\lambda,\theta)$. Indeed, by applying the Lindley method, $I(\boldsymbol{X})$ can be rewritten as

$$\begin{aligned} I(\boldsymbol{X}) = u\left(\hat{\lambda},\hat{\theta}\right) &+ \frac{1}{2}\left[\left(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\hat{\rho}_\lambda\right)\hat{\sigma}_{\lambda\lambda} + \left(\hat{u}_{\theta\lambda} + 2\hat{u}_\theta\hat{\rho}_\lambda\right)\hat{\sigma}_{\theta\lambda}\right. \\ &\left.+\left(\hat{u}_{\lambda\theta} + 2\hat{u}_\lambda\hat{\rho}_\theta\right)\hat{\sigma}_{\lambda\theta} + \left(\hat{u}_{\theta\theta} + 2\hat{u}_\theta\hat{\rho}_\theta\right)\hat{\sigma}_{\theta\theta}\right] \\ &+ \frac{1}{2}\left[\left(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{u}_\theta\hat{\sigma}_{\lambda\theta}\right)\left(\hat{l}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{l}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\right)\right. \\ &\left.+\left(\hat{u}_\lambda\hat{\sigma}_{\theta\lambda} + \hat{u}_\theta\hat{\sigma}_{\theta\theta}\right)\left(\hat{l}_{\theta\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{l}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\right)\right], \end{aligned}$$

where $\hat{\lambda}$ and $\hat{\theta}$ are the MLEs of λ and θ, respectively. Also, $u_{\lambda\lambda}$ is the second derivative of the function $u(\lambda,\theta)$ with respect to λ and $\hat{u}_{\lambda\lambda}$ is the second derivative of the function $u(\lambda,\theta)$ with respect to λ evaluated at $\left(\hat{\lambda},\hat{\theta}\right)$.Also, $\sigma_{ij} = (i,j)$th elements of the inverse of the matrix $\left[-\frac{\partial^2 l(\lambda,\theta|X)}{\partial\lambda\partial\theta}\right]^{-1}$ are evaluated at $\left(\hat{\lambda},\hat{\theta}\right)$. Also expressions of $l_{\lambda\lambda}, l_{\theta\theta}, l_{\theta\lambda}, l_{\lambda\lambda\lambda}, l_{\theta\lambda\lambda}$ and $l_{\theta\theta\theta}$ are presented in "Appendix."

For the squared error loss function L_{SB}, we get that

$$u(\lambda,\theta)=\lambda, u_\lambda=1, \quad \text{and} \quad u_{\lambda\lambda}=u_\theta=u_{\theta\theta}=u_{\theta\lambda}=u_{\lambda\theta}=0,$$

and the corresponding Bayesian estimate of λ is

$$\hat{\alpha}_{\text{SB}}=E(\lambda|X)=\hat{\lambda}+0.5\left[2\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+2\hat{\rho}_\theta\hat{\sigma}_{\lambda\theta}+\hat{\sigma}^2_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\lambda}+2\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\theta}\right].$$

Next, the Bayesian estimate of θ under L_{SB} is obtained as
(Here $u(\lambda,\theta)=\theta, u_\theta=1$, and $u_\lambda=u_{\lambda\lambda}=u_{\theta\theta}=u_{\theta\lambda}=u_{\lambda\theta}=0$)

$$\hat{\theta}_{\text{SB}}=E(\theta|X)=\hat{\theta}+0.5\left[2\hat{\rho}_\theta\hat{\sigma}_{\theta\theta}+2\hat{\rho}_\lambda\hat{\sigma}_{\theta\lambda}+\hat{\sigma}^2_{\theta\theta}\hat{l}_{\theta\theta\theta}+3\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\lambda\lambda}\right].$$

For the loss function L_{LB}, noticing that in this case we have

$$u(\lambda,\theta)=\mathrm{e}^{-h\lambda}, \quad u_\lambda=-h\mathrm{e}^{-h\lambda}, \quad u_{\lambda\lambda}=h^2\mathrm{e}^{-h\lambda}, \quad \text{and} \quad u_\theta=u_{\theta\theta}=u_{\theta\lambda}=u_{\lambda\theta}=0,$$

and with

$$E\left(\mathrm{e}^{-h\lambda}|\boldsymbol{x}\right)=\mathrm{e}^{-h\hat{\lambda}}+0.5\left[\hat{u}_{\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{u}_\lambda\left(2\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+2\hat{\rho}_\theta\hat{\sigma}_{\lambda\theta}+\hat{\sigma}^2_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\lambda}\right.\right.$$
$$\left.\left.+2\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\theta}\right)\right],$$

the Bayesian estimate of λ is obtained as

$$\hat{\lambda}_{\text{LB}}=-\frac{1}{h}\ln\left\{E\left(\mathrm{e}^{-h\lambda}|\boldsymbol{x}\right)\right\}.$$

Similarly, for θ we have

$$u(\lambda,\theta)=\mathrm{e}^{-h\theta}, \quad u_\theta=-h\mathrm{e}^{-h\theta}, \quad u_{\theta\theta}=h^2\mathrm{e}^{-h\theta}, \quad \text{and} \quad u_\lambda=u_{\lambda\lambda}=u_{\theta\lambda}=u_{\lambda\theta}=0,$$
$$E\left(\mathrm{e}^{-h\theta}|\boldsymbol{x}\right)=\mathrm{e}^{-h\hat{\theta}}+0.5\left[\hat{u}_{\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{u}_\theta\left(2\hat{\rho}_\theta\hat{\sigma}_{\theta\theta}+2\hat{\rho}_\lambda\hat{\sigma}_{\theta\lambda}+\hat{\sigma}^2_{\theta\theta}\hat{l}_{\theta\theta\theta}+3\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\lambda\lambda}\right)\right],$$
$$\hat{\theta}_{\text{LB}}=-\frac{1}{h}\ln\left\{E\left(\mathrm{e}^{-h\theta}|\boldsymbol{x}\right)\right\}.$$

Finally, we consider the ENTROPY loss function. Notice that for the parameter λ and loss function L_{EB},

$$u(\lambda,\theta)=\lambda^{-w}, \quad u_\lambda=-w\lambda^{-(w+1)}, \quad u_{\lambda\lambda}=w(w+1)\lambda^{-(w+2)},$$
$$\text{and} \quad u_\theta=u_{\theta\theta}=u_{\theta\lambda}=u_{\lambda\theta}=0,$$
$$E(\lambda^{-w}|\boldsymbol{x})=\hat{\lambda}^{-w}+0.5\left[\hat{u}_{\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{u}_\lambda\left(2\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+2\hat{\rho}_\theta\hat{\sigma}_{\lambda\theta}+\hat{\sigma}^2_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\lambda}\right.\right.$$
$$\left.\left.+2\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\theta\theta\theta}\right)\right].$$

Thus, the approximate Bayesian estimate of λ in this case is given by

$$\hat{\lambda}_{\text{EB}}=\{E(\lambda^{-w}|\boldsymbol{x})\}^{-\frac{1}{w}}.$$

Also, for the parameter θ we get that

$$u(\lambda,\theta)=\theta^{-w}, \quad u_\theta=-w\theta^{-(w+1)}, \quad u_{\theta\theta}=w(w+1)\theta^{-(w+2)}, \quad \text{and} \quad u_\lambda=u_{\lambda\lambda}=u_{\theta\lambda}=u_{\lambda\theta}=0,$$
$$E(\theta^{-w}|\boldsymbol{x})=\hat{\theta}^{-w}+0.5\left[\hat{u}_{\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{u}_\theta\left(2\hat{\rho}_\theta\hat{\sigma}_{\theta\theta}+2\hat{\rho}_\lambda\hat{\sigma}_{\theta\lambda}+\hat{\sigma}^2_{\theta\theta}\hat{l}_{\theta\theta\theta}+3\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\hat{l}_{\lambda\theta\theta}+\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{l}_{\lambda\lambda\lambda}\right)\right].$$

Consequently,

$$\hat{\theta}_{\mathrm{EB}} = \{E(\theta^{-w}|\boldsymbol{x})\}^{-\frac{1}{w}}.$$

3.3 Metropolis Hasting Algorithm

Metropolis–Hastings (M–H) algorithm is a useful method for generating random samples from the posterior distribution using a proposal density. Let $g(.)$ be the density of the proposal distribution. Since the support of the parameters of our distribution is positive, we consider the chi -square distribution as our proposal density for estimating the posterior samples from λ. We also consider the standard uniform distribution as candidate distribution for θ. Based on (3.4), the posterior distribution of λ and θ for the given sample x is as follows:

$$\pi(\lambda|x) = k^{-1}(x)\lambda^{m+a_1-1}\mathrm{e}^{-\lambda\left(\sum_{j=1}^{m} x_j+b_1\right)} \times \int_0^1 \theta^{m+a_2-1} \times (1-\theta)^{b_2-1}$$
$$\exp\left\{-2\sum_{j=1}^{m} \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right) + \sum_{j=1}^{m} R_j \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right)\right\}\mathrm{d}\theta,$$

and

$$\pi(\theta|x) = k^{-1}(x)\theta^{m+a_2-1}(1-\theta)^{b_2-1} \times \int_0^1 \lambda^{m+a_1-1}\mathrm{e}^{-\lambda\left(\sum_{j=1}^{m} x_j+b_1\right)}$$
$$\times \exp\left\{-2\sum_{j=1}^{m} \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right) + \sum_{j=1}^{m} R_j \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right)\right\}\mathrm{d}\lambda,$$

where

$$k(x) = \int_0^\infty \int_0^1 \lambda^{m+a_1-1}\mathrm{e}^{-\lambda\left(\sum_{j=1}^{m} x_j+b_1\right)}\theta^{m+a_2-1}(1-\theta)^{b_2-1}$$
$$\times \exp\left\{-2\sum_{j=1}^{m} \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right) + \sum_{j=1}^{m} R_j \ln\left(\mathrm{e}^{-\lambda x_j}(1-\theta)+\theta\right)\right\}\mathrm{d}\lambda\mathrm{d}\theta.$$

It is clear that both posterior distributions do not have closed form; therefore, we use the Metropolis–Hasting algorithm to obtain our Bayes estimators based on posterior samples, suppose the $\pi(\lambda|x)$ is the posterior distribution of the MH algorithm steps as follows:

Given $\lambda^{(t)}$,

1. Generate $Y_t \sim g(y)$

2. Take $\lambda^{(t+1)} = Y_t$ with probability $p = \min\left\{1, \frac{\pi(Y_t|x)}{\pi(\lambda^{(t)}|x)} \cdot \frac{g(\lambda^{(t)})}{g(Y_t)}\right\}$ and $\lambda^{(t+1)} = \lambda^{(t)}$ with probability $1 - p$,

where $g(.)$ is the p.d.f of a chi-square distribution with four degrees of freedom. With a similar approach, the M–H samples can be drawn from the posterior distribution of $\theta|x$ with the standard uniform as a proposal distribution. Finally, from the random sample of size M drawn from the posterior density, some of the initial samples (burn-in) can be discarded, and the remaining samples can be further utilized to compute Bayes estimates. More precisely, the Bayes estimators of any function $g(\theta, \lambda)$ of parameters can be given

$$\hat{g}_{\mathrm{MH}}(\lambda, \theta) = \frac{1}{M - l_{\mathrm{B}}} \sum_{i=l_{\mathrm{B}}}^{M} g(\lambda_i, \theta_i).$$

Here l_{B} represents the number of burn-in samples. Next, we will use the method of Chen and Shao [8] to obtain HPD interval estimates for the unknown parameters of the CEG distribution. This method has been extensively used for constructing HPD intervals for the unknown parameters of the distribution of interest. In the literature, samples drawn from the posterior density using importance sampling technique are used to construct HPD intervals, see Dey and Dey [13], Kundu and Pradhan [21] and Singh et al. [39]. In the present work, we will utilize the samples drawn using the proposed MH algorithm to construct the interval estimates [11]. More precisely, let us suppose that $\pi(\theta|x)$ denotes the posterior distribution function of θ. Let us further assume that $\theta^{(p)}$ be the pth quantile of θ, that is, $\theta^{(p)} = \inf\{\theta : \pi(\theta|x) \geq p\}$, where $0 < p < 1$. Observe that for a given θ^*, a simulation consistent estimator of $\pi(\theta^*|x)$ can be obtained as

$$\pi(\theta^*|x) = \frac{1}{M - l_{\mathrm{B}}} \sum_{i=l_{\mathrm{B}}}^{M} I_{\theta \leq \theta^*}.$$

Here $I_{\theta \leq \theta^*}$ is the indicator function. Then the corresponding estimate is obtained as

$$\hat{\pi}(\theta^*|x) = \begin{cases} 0, & \text{if } \theta^* < \theta_{(l_{\mathrm{B}})} \\ \sum_{j=l_{\mathrm{B}}}^{M} \omega_j, & \text{if } \theta_{(i)} < \theta^* < \theta_{(i+1)} \\ 1, & \text{if } \theta^* > \theta_{(M)} \end{cases}$$

where $\omega_j = \frac{1}{M - l_{\mathrm{B}}}$ and $\theta_{(j)}$ are the ordered values of θ_j. Now, for $i = l_{\mathrm{B}}, \ldots, M$, $\theta^{(p)}$ can be approximated by

$$\hat{\theta}^{(p)} = \begin{cases} \theta_{(l_B)}, & \text{if } p = 0, \\ \theta_{(i)}, & \text{if } \sum_{j=l_{\mathrm{B}}}^{i-1} \omega_j < p < \sum_{j=l_{\mathrm{B}}}^{i} \omega_j. \end{cases}$$

Now to obtain a $100(1-p)\%$ HPD credible interval for θ, let $Rj = \left(\hat{\theta}^{\left(\frac{j}{M}\right)}, \hat{\theta}^{\left(\frac{j+(1-p)M}{M}\right)}\right)$ for $j = l_B, \ldots, [pM]$, here $[a]$ denotes the largest integer less than or equal to a. Then choose R_{j*} among all the $R_j's$ such that it has the smallest width.

4 Simulation Study and Illustrative Example

In this section, we conduct some simulation study to compare the performance of the different methods proposed in the previous sections. For hyper-parameters of prior distributions, we set $a_1 = b_1 = 0$ and $a_2 = b_2 = 0$. Further, it is supposed that $h = q = 1$ in the LINEX and ENTROPY loss functions. In this importance sampling method, we generate 1000 MCMC samples and calculated the related Bayes estimators while in M–H algorithm we generate 10,000 samples and withdraw the first 5000 and then obtain the related Bayes estimates based on the remaining samples. The acceptance rate for M–H algorithm 0.513 with DIC = 50.81586 which are reasonable values. Note that in generating the M–H sample we use the MLE's of λ and $\theta\left(\lambda^{(0)}, \theta^{(0)}\right) = \left(\hat{\lambda}_{\mathrm{MLE}}, \hat{\theta}_{\mathrm{MLE}}\right)$ as the initial values Markov chains.

We considered three different censoring schemes in Table 1. We run the whole process for 10,000 times, and the results are provided in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 and 21 for different values of the parameters. Note that, for the shrinkage estimators we used the relative efficiencies formula as follows.

$$\mathrm{RE}\left(\hat{\theta}^{\mathrm{SPT}}_{\mathrm{Bayes}}, \hat{\theta}_{\mathrm{Bayes}}\right) = \frac{\mathrm{MSE}\left(\hat{\theta}_{\mathrm{Bayes}}\right)}{\mathrm{MSE}\left(\hat{\theta}^{\mathrm{SPT}}_{\mathrm{Bayes}}\right)},$$

where we use $\frac{1}{n}\sum_{j=1}^{n}\left(\hat{\theta}^{j}_{\mathrm{SEL}} - \theta\right)^2$, $\frac{1}{n}\sum_{j=1}^{n}[\mathrm{e}^{h\left(\hat{\theta}^{j}_{\mathrm{Linex}} - \theta\right)} - h\left(\hat{\theta}^{j}_{\mathrm{Linex}} - \theta\right) - 1]$, $\frac{1}{n}\sum_{j=1}^{n}[\left(\frac{\hat{\theta}^{j}_{\mathrm{Entropy}}}{\theta}\right)^{q} - q\ln\left(\frac{\hat{\theta}^{i}_{\mathrm{Entropy}}}{\theta}\right) - 1]$, for the estimated risk values of the Bayes estimators based on SEL, LINEX and ENTROPY loss functions. We generate the censored data from CEG distribution with parameters $\lambda = 2,\ \theta = 0.5$ and $\lambda = 5,\ \theta = 0.6$. The results for the NR and EM methods are shown in Tables 2 and 3. Further the estimated risk and biases for the Bayes estimators for different loss functions are provided in Tables 5 and 6, next in Tables 10, 11, 12 and 13 the simulations results are given for the shrinkage estimators. Note, in Table 10, we assume the prior guesses to $\lambda_0 = 2.2$, $\theta_0 = 0.6$ and in Table 11, we take $\lambda_0 = 5.2$, $\theta_0 = 0.7$. Further the simulated Pitman closeness (PC) for comparing the EM and NR methods is as follows.

$$\mathrm{PC} = P\left\{\left|\hat{\vartheta}_{\mathrm{EM}} - \vartheta\right| < \left|\hat{\vartheta}_{\mathrm{NR}} - \vartheta\right|\right\} = \frac{1}{1000}\#\left\{\left|\hat{\vartheta}_{\mathrm{EMi}} - \vartheta\right| < \left|\hat{\vartheta}_{\mathrm{NRi}} - \vartheta\right|\right\}.$$

We say that $\hat{\vartheta}_{\mathrm{EM}}$ competes with $\hat{\vartheta}_{\mathrm{NR}}$ if > 0.5.

Table 1 Censoring scheme $R = (r_1, \ldots, r_m)$

n	m	Scheme	Scheme
30	20	1	$(10, 0^{*19})$
		2	$(1, 2, 1, 3, 3, 0^{*15})$
		3	$(0, 1, 0^{*4}, 2, 0^{*3}, 2, 0^{*2}, 3, 0^{*2}, 1, 0^{*2}, 1)$
50	35	4	$(15, 0^{*34})$
		5	$(0^{*34}, 15)$
		6	$(0, 1, 0^{*2}, 2, 0^{*4}, 2, 0^{*2}, 1, 0^{*2}, 2, 0^{*4}, 1, 0^{*2}, 1, 0^{*4}, 1, 0^{*2}, 2, 0, 1, 0)$
100	80	7	$(20, 0^{*79})$
		8	$(0^{*79}, 20)$
		9	$(0^{*19}, 5, 0^{*19}, 5, 0^{*19}, 5, 0^{*19}, 5)$

Table 2 Bias and MSE (in parentheses) of the estimators with $\lambda = 2$ and $\theta = 0.5$

Scheme	NR		EM	
	λ	θ	λ	θ
1	0.2585 (**0.7809**)	0.0756 (**0.2235**)	0.0808 (**0.2586**)	0.0240 (**0.0485**)
2	0.2788 (**0.8303**)	0.0628 (**0.1832**)	0.1524 (**0.3363**)	0.0070 (**0.0361**)
3	0.3274 (**1.0734**)	0.0959 (**0.3112**)	0.2015 (**0.4889**)	0.0103 (**0.0578**)
4	0.1525 (**0.3890**)	0.0440 (**0.1116**)	0.0346 (**0.1085**)	0.0070 (**0.0113**)
5	0.2295 (**0.8090**)	0.1506 (**0.7161**)	0.0722 (**0.1544**)	0.0004 (**0.0177**)
6	0.5301 (**0.8351**)	− 0.0524 (**0.0774**)	0.2314 (**0.2102**)	0.0039 (**0.0150**)
7	0.0719 (**0.1433**)	0.0145 (**0.0383**)	0.0079 (**0.0412**)	0.0097 (**0.0034**)
8	0.0731 (**0.2235**)	0.0379 (**0.0668**)	0.0226 (**0.0435**)	0.0050 (**0.0046**)
9	0.0658 (**0.1835**)	0.0278 (**0.0480**)	0.0120 (**0.0458**)	0.0069 (**0.0047**)

Table 3 Bias and MSE (in parentheses) of the estimators with $\lambda = 5$ and $\theta = 0.6$

Scheme	NR		EM	
	λ	θ	λ	θ
1	0.7160 (**5.5910**)	0.0905 (**0.3306**)	0.3280 (**2.7502**)	0.0287 (**0.0837**)
2	0.7736 (**5.9825**)	0.0745 (**0.2695**)	0.5174 (**3.5906**)	0.0340 (**0.1354**)
3	0.9205 (**7.9352**)	0.1216 (**0.5041**)	0.7759 (**5.7106**)	0.0128 (**0.1421**)
4	0.4220 (**2.7578**)	0.0525 (**0.1635**)	0.1653 (**1.0516**)	0.0171 (**0.0367**)
5	0.6450 (**6.0999**)	0.2980 (**5.1355**)	0.1609 (**1.8422**)	0.0283 (**0.0331**)
6	1.4657 (**6.2038**)	− 0.0686 (**0.1129**)	0.7299 (**2.4367**)	0.0128 (**0.0491**)
7	0.1969 (**1.0030**)	0.0172 (**0.0560**)	0.0352 (**0.3594**)	0.0099 (**0.0166**)
8	0.2059 (**1.6423**)	0.0498 (**0.1079**)	0.1640 (**0.5431**)	− 0.0127 (**0.0156**)
9	0.1860 (**1.3184**)	0.0343 (**0.0730**)	0.1150 (**0.4875**)	− 0.0031 (**0.0146**)

Table 4 PC comparison of MLEs based on EM and NR algorithms

λ	θ	Scheme 1	Scheme 3	Scheme 4	Scheme 5	Scheme 7	Scheme 9
PC for $\hat{\lambda}_{\mathrm{EM}}$ versus $\hat{\lambda}_{\mathrm{NR}}$							
2	0.5	0.7550	0.8050	0.7760	0.8090	0.7340	0.7800
5	0.6	0.7810	0.8180	0.7830	0.8140	0.7820	0.8190
PC for $\hat{\theta}_{\mathrm{EM}}$ versus $\hat{\theta}_{\mathrm{NR}}$							
2	0.5	0.9480	0.9640	0.9640	0.9590	0.9530	0.9650
5	0.6	0.8780	0.8570	0.8720	0.8610	0.9510	0.9440

This simulation results reveal that the SEM is always superior to the NR method in terms of estimated biases, MSE's. Further it is seen that SEM estimated is Pitman closer to the parameters than to the Bayes NR estimates. We also observe that the shrinkage Bayes estimated have smaller estimated risk than the usual Bayes estimated based on MCMC method. It is shown that the relative efficacies of the proposed shrinkage estimated are higher than 1 which is indicated to use of the shrinkages estimators in the case of having suspected non-sample prior information. We also observe that the Bayes estimators based on M–H algorithm, mostly, perform those based on Lindely approximation and MCMC method.

4.1 Real Data Analysis

For illustrative purposes, here real data are analyzed using the proposed methods. A data set on the endurance of deep groove ball bearings analyzed by Lieblein and Zelen [23] consists of the number of million revolutions before failure for each of 23 ball bearings used in a life test. The data set is as follows.

17.88	42.12	51.96	68.64	93.12	127.96
28.92	45.60	54.12	68.64	98.64	128.01
33.00	48.48	55.56	68.88	105.12	173.4
41.52	51.84	67.80	84.12	105.84	

Louzada et al. [26] indicated that the CEG can be fitted to this data set quite well. For our purpose, we generate three different schemes of progressive type-II censored sample as follows.

Scheme 1: $R=(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12)$
Scheme 2: $R=(12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
Scheme 3: $R=(0, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0)$

The estimated values of the parameters are given in the following Tables 22, 23, 24 and 25 while the approximate and Bayesian confidence intervals are presented in Table 26.

Table 5 Bias and estimated risk (in parentheses) of the Bayes estimators with $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	−0.4328 (0.2669)	0.3185 (0.1046)	−0.4806 (0.1226)	0.3144 (0.0572)	−0.4956 (0.0525)	0.3087 (0.1394)
2	−0.4457 (0.2914)	0.3188 (0.1048)	−0.4932 (0.1314)	0.3148 (0.0573)	−0.5081 (0.0571)	0.3090 (0.1396)
3	−0.4910 (0.3331)	0.3231 (0.1074)	−0.5364 (0.1485)	0.3190 (0.0588)	−0.5524 (0.0657)	0.3133 (0.1428)
4	−0.4721 (0.2764)	0.3812 (0.1468)	−0.4988 (0.1221)	0.3796 (0.0832)	−0.5075 (0.0501)	0.3776 (0.1937)
5	−0.5771 (0.3767)	0.3849 (0.1495)	−0.6003 (0.1609)	0.3832 (0.0848)	−0.6103 (0.0694)	0.3811 (0.1966)
6	−0.4537 (0.2613)	0.3817 (0.1471)	−0.4810 (0.1162)	0.3800 (0.0833)	−0.4895 (0.0471)	0.3780 (0.1940)
7	−0.5178 (0.2914)	0.4411 (0.1950)	−0.5289 (0.1250)	0.4407 (0.1134)	−0.5329 (0.0486)	0.4403 (0.2493)
8	−0.6005 (0.3799)	0.4424 (0.1961)	−0.6105 (0.1588)	0.4420 (0.1141)	−0.6149 (0.0648)	0.4416 (0.2505)
9	−0.5718 (0.3498)	0.4419 (0.1957)	−0.5821 (0.1472)	0.4415 (0.1139)	−0.5863 (0.0593)	0.4411 (0.2500)

Table 6 Bias and estimated risk (in parentheses) of the Bayes estimators with $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	− 1.2142 (1.8648)	0.2337 (0.0574)	− 1.4947 (0.7565)	0.2294 (0.0302)	− 1.3732 (0.0595)	0.2233 (0.0583)
2	− 1.2313 (1.9701)	0.2344 (0.0577)	− 1.5118 (0.7756)	0.2301 (0.0304)	− 1.3902 (0.0629)	0.2240 (0.0586)
3	− 1.3008 (2.1518)	0.2392 (0.0598)	− 1.5764 (0.8278)	0.2350 (0.0316)	− 1.4592 (0.0689)	0.2289 (0.0607)
4	− 1.1259 (1.5881)	0.2878 (0.0842)	− 1.2987 (0.6149)	0.2860 (0.0460)	− 1.2185 (0.0460)	0.2839 (0.0867)
5	− 1.3117 (1.9981)	0.2917 (0.0864)	− 1.4700 (0.7318)	0.2900 (0.0473)	− 1.4007 (0.0585)	0.2878 (0.0887)
6	− 1.0278 (1.3950)	0.2887 (0.0847)	− 1.2103 (0.5587)	0.2870 (0.0463)	− 1.1230 (0.0401)	0.2848 (0.0872)
7	− 1.0968 (1.3655)	0.3430 (0.1180)	− 1.1744 (0.5091)	0.3426 (0.0663)	− 1.1372 (0.0359)	0.3422 (0.1193)
8	− 1.2629 (1.7355)	0.3442 (0.1188)	− 1.3350 (0.6170)	0.3438 (0.0668)	− 1.3021 (0.0462)	0.3433 (0.1200)
9	− 1.1987 (1.6006)	0.3437 (0.1185)	− 1.2726 (0.5763)	0.3433 (0.0666)	− 1.2381 (0.0425)	0.3429 (0.1197)

Table 7 PC comparison of MLEs based on EM and Bayes (under SEL function) algorithms

λ	θ	Scheme 1	Scheme 3	Scheme 4	Scheme 5	Scheme 7	Scheme 9
PC for $\hat{\lambda}_{\text{EM}}$ versus $\hat{\lambda}_{\text{Bayes(SEL)}}$							
2	0.5	0.6970	0.6830	0.7810	0.8470	0.9330	0.9400
5	0.6	0.6800	0.6280	0.7760	0.7850	0.8610	0.8610
PC for $\hat{\theta}_{\text{EM}}$ versus $\hat{\theta}_{\text{Bayes(SEL)}}$							
2	0.5	0.9430	0.9200	0.9920	0.9910	1	0.9980
5	0.6	0.7460	0.6440	0.9040	0.8990	0.9810	0.9720

Table 8 PC comparison of MLEs based on EM and Bayes (under LINEX loss function) algorithms

λ	θ	Scheme 1	Scheme 3	Scheme 4	Scheme 5	Scheme 7	Scheme 9
PC for $\hat{\lambda}_{\text{EM}}$ versus $\hat{\lambda}_{\text{Bayes(LINEX)}}$							
2	0.5	0.7210	0.6990	0.7990	0.8590	0.9370	0.9430
5	0.6	0.7460	0.6770	0.8160	0.8110	0.8830	0.8810
PC for $\hat{\theta}_{\text{EM}}$ versus $\hat{\theta}_{\text{Bayes(LINEX)}}$							
2	0.5	0.9420	0.9160	0.9920	0.9900	1	0.9980
5	0.6	0.7430	0.6350	0.9020	0.8980	0.9810	0.9720

Table 9 PC comparison of MLEs based on EM and Bayes (under ENTROPY loss function) algorithms

λ	θ	Scheme 1	Scheme 3	Scheme 4	Scheme 5	Scheme 7	Scheme 9
PC for $\hat{\lambda}_{\text{EM}}$ versus $\hat{\lambda}_{\text{Bayes(ENTROPY)}}$							
2	0.5	0.7280	0.7050	0.8010	0.8600	0.9370	0.9460
5	0.6	0.7160	0.6520	0.7980	0.7980	0.8710	0.8690
PC for $\hat{\theta}_{\text{EM}}$ versus $\hat{\theta}_{\text{Bayes(ENTROPY)}}$							
2	0.5	0.9390	0.9130	0.9910	0.9900	1	0.9980
5	0.6	0.7340	0.6230	0.9000	0.8970	0.9810	0.9720

5 Summary and Conclusion

In this paper, we proposed different estimators for the parameters of the complementary exponential distribution. We obtained maximum likelihood estimators based on N–R and stochastic expectation maximization method as well. Further different sorts of Bayes estimates are obtained under various loss functions. We

Table 10 Bias and estimated risk (in parentheses) of the shrinkage estimators with $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	−0.2829 (0.1641)	0.2696 (0.0773)	−0.3190 (0.0750)	0.2664 (0.0418)	−0.3302 (0.0317)	0.2619 (0.1066)
2	−0.1459 (0.0603)	0.2350 (0.0583)	−0.1710 (0.0286)	0.2326 (0.0312)	−0.1790 (0.0106)	0.2290 (0.0837)
3	−0.1455 (0.0442)	0.2115 (0.0455)	−0.1682 (0.0224)	0.2095 (0.0240)	−0.1762 (0.0071)	0.2066 (0.0682)
4	−0.3527 (0.1962)	0.3322 (0.1157)	−0.3743 (0.0866)	0.3309 (0.0650)	−0.3815 (0.0355)	0.3292 (0.1565)
5	−0.1885 (0.0465)	0.2424 (0.0591)	−0.2001 (0.0231)	0.2416 (0.0319)	−0.2051 (0.0072)	0.2406 (0.0887)
6	−0.1269 (0.0299)	0.2408 (0.0584)	−0.1405 (0.0152)	0.2400 (0.0315)	−0.1448 (0.0046)	0.2390 (0.0877)
7	−0.4120 (0.2187)	0.3937 (0.1611)	−0.4213 (0.0940)	0.3934 (0.0930)	−0.4247 (0.0364)	0.3930 (0.2101)
8	−0.2002 (0.0449)	0.2712 (0.0736)	−0.2052 (0.0216)	0.2710 (0.0403)	−0.2074 (0.0065)	0.2708 (0.1089)
9	−0.1859 (0.0403)	0.2709 (0.0735)	−0.1911 (0.0195)	0.2708 (0.0403)	−0.1932 (0.0058)	0.2706 (0.1087)

Table 11 Bias and estimated risk (in parentheses) of the shrinkage estimators with $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	−0.8724 (1.1618)	0.2025 (0.0437)	−1.0842 (0.4846)	0.1992 (0.0228)	−0.9923 (0.0363)	0.1945 (0.0456)
2	−0.5621 (0.5027)	0.1819 (0.0348)	−0.7108 (0.2364)	0.1792 (0.0181)	−0.6463 (0.0144)	0.1755 (0.0374)
3	−0.5504 (0.4179)	0.1696 (0.0294)	−0.6882 (0.2130)	0.1675 (0.0152)	−0.6296 (0.0113)	0.1645 (0.0325)
4	−0.8609 (1.0994)	0.2528 (0.0669)	−0.9995 (0.4318)	0.2514 (0.0363)	−0.9350 (0.0316)	0.2497 (0.0703)
5	−0.5558 (0.3783)	0.1959 (0.0387)	−0.6350 (0.1815)	0.1950 (0.0205)	−0.6004 (0.0095)	0.1939 (0.0434)
6	−0.4139 (0.2560)	0.1943 (0.0381)	−0.5051 (0.1311)	0.1935 (0.0202)	−0.4615 (0.0064)	0.1924 (0.0428)
7	−0.8795 (0.9979)	0.3078 (0.0981)	−0.9434 (0.3780)	0.3075 (0.0548)	−0.9128 (0.0260)	0.3071 (0.1006)
8	−0.5315 (0.3176)	0.2221 (0.0494)	−0.5675 (0.1440)	0.2219 (0.0266)	−0.5510 (0.0074)	0.2217 (0.0551)
9	−0.4993 (0.2903)	0.2219 (0.0493)	−0.5363 (0.1327)	0.2217 (0.0266)	−0.5190 (0.0068)	0.2215 (0.0550)

Table 12 Relative efficiencies (RE) of Bayesian shrinkage estimates with respect to the Bayes estimates with $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	1.6264	1.3532	1.6347	1.3684	1.6562	1.3077
2	4.8325	1.7976	4.5944	1.8365	5.3868	1.6679
3	7.5362	2.3604	6.6295	2.4500	9.2535	2.0938
4	1.4088	1.2688	1.4099	1.2800	1.4113	1.2377
5	8.1011	2.5296	6.9654	2.6583	9.6389	2.2165
6	8.7391	2.5188	7.6447	2.6444	10.2391	2.2121
7	1.3324	1.2104	1.3298	1.2194	1.3352	1.1866
8	8.4610	2.6644	7.3519	2.8313	9.9692	2.3003
9	8.6799	2.6626	7.5487	2.8263	10.2241	2.2999

Table 13 Relative efficiencies (RE) of Bayesian shrinkage estimates with respect to the Bayes estimates with $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	1.6051	1.3135	1.5611	1.3246	1.6391	1.2785
2	3.9190	1.6580	3.2809	1.6796	4.3681	1.5668
3	5.1491	2.0340	3.8864	2.0789	6.0973	1.8677
4	1.4445	1.2586	1.4240	1.2672	1.4557	1.2333
5	5.2818	2.2326	4.0320	2.3073	6.1579	2.0438
6	5.4492	2.2231	4.2616	2.2921	6.2656	2.0374
7	1.3684	1.2029	1.3468	1.2099	1.3808	1.1859
8	5.4644	2.4049	4.2847	2.5113	6.2432	2.1779
9	5.5136	2.4037	4.3429	2.5038	6.2500	2.1764

also proposed the shrinkage estimators which has higher relative efficiency than the usual Bayes estimates. The Bayesian credible intervals are also computed by means of MCMC samples. We found that maximum likelihood estimators of the unknown parameters of the distribution do not admit closed form, and further the EM algorithm for this purpose still requires optimization technique to solve the involved expressions. Therefore, we considered the SEM algorithm to obtain the maximum likelihood estimators. In simulation study, we presented a comparison between the estimates obtained using SEM algorithm and estimates from Newton–Raphson and EM algorithm. We observed that the performance of SEM algorithm is quite satisfactory. For illustration purpose, we also considered a real data set. It should be mentioned here that the prediction of the future-order statistics based on the progressive type-II censored samples is also in progress by the authors and we hope to report these results in another communication.

Table 14 Bias and estimated risk (in parentheses) of the Lindley estimators with $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	0.9940 (1.4498)	−0.0237 (0.1788)	0.3522 (0.6861)	−0.0730 (0.1771)	0.6477 (1.2602)	−0.2068 (0.1709)
2	1.1899 (1.5371)	−0.0113 (0.1693)	0.4026 (0.7806)	−0.0653 (0.1728)	0.6427 (1.1852)	−0.2233 (0.1706)
3	0.8802 (1.0576)	−0.0686 (0.1220)	0.4282 (0.5463)	−0.0897 (0.1215)	0.6784 (1.1205)	−0.1474 (0.1223)
4	0.7657 (1.2828)	−0.0391 (0.1370)	0.2708 (0.5358)	−0.0738 (0.1425)	0.4862 (1.1282)	−0.1837 (0.1598)
5	0.1501 (0.0918)	−0.0123 (0.0316)	0.0755 (0.1099)	−0.0200 (0.0312)	0.0748 (0.1045)	−0.0400 (0.0278)
6	0.7007 (0.8937)	−0.0550 (0.1018)	0.4458 (0.5171)	−0.0724 (0.1066)	0.5752 (0.8559)	−0.1143 (0.1095)
7	0.5646 (1.2039)	−0.0351 (0.1126)	0.1959 (0.4489)	−0.0543 (0.1172)	0.3338 (0.8461)	−0.1371 (0.1539)
8	0.0607 (0.0312)	−0.0009 (0.0095)	0.0273 (0.0300)	−0.0051 (0.0101)	0.0273 (0.0300)	−0.0173 (0.0110)
9	0.0797 (0.0765)	−0.0052 (0.0325)	0.0438 (0.1071)	−0.0114 (0.0317)	0.0429 (0.0923)	−0.0283 (0.0284)

Table 15 Bias and estimated risk (in parentheses) of the Lindley estimators with $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	1.4031 (1.1675)	−0.0550 (0.2314)	−0.1532 (0.8342)	−0.1029 (0.2055)	0.9165 (1.2086)	−0.2483 (0.2041)
2	1.6016 (1.1159)	−0.0455 (0.2172)	−0.2935 (0.7093)	−0.1197 (0.2128)	0.9875 (1.2077)	−0.2721 (0.1995)
3	2.0788 (1.0409)	−0.1015 (0.1665)	0.1033 (0.6786)	−0.1457 (0.1736)	1.4898 (1.1349)	−0.2348 (0.1707)
4	1.1925 (1.1882)	−0.0641 (0.1881)	0.0067 (0.7589)	−0.1108 (0.1864)	0.7850 (1.1195)	−0.2424 (0.2010)
5	0.4776 (0.3181)	0.0053 (0.0520)	−0.0976 (0.3247)	−0.0083 (0.0502)	0.2640 (0.3639)	−0.0387 (0.0512)
6	1.6841 (0.8989)	−0.0861 (0.1359)	0.4294 (0.6851)	−0.1124 (0.1414)	1.3957 (0.9901)	−0.1667 (0.1477)
7	0.7197 (0.9692)	−0.0555 (0.1399)	0.0458 (0.5179)	−0.0830 (0.1457)	0.4966 (0.9004)	−0.1826 (0.1899)
8	0.2313 (0.1225)	−0.0160 (0.0184)	−0.0120 (0.1227)	−0.0217 (0.0185)	0.1378 (0.1240)	−0.0339 (0.0178)
9	0.3581 (0.2922)	−0.0229 (0.0436)	0.0545 (0.2139)	−0.0317 (0.0443)	0.2423 (0.2288)	−0.0483 (0.0378)

Table 16 Bias and estimated risk (in parentheses) of the M–H algorithm with $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	−0.0477 (0.1608)	0.1046 (0.0248)	−0.1743 (0.0685)	0.0821 (0.0111)	−0.1814 (0.0231)	0.0069 (0.0488)
2	−0.0381 (0.1603)	0.1032 (0.0236)	−0.1700 (0.0671)	0.0809 (0.0105)	−0.1770 (0.0229)	0.0073 (0.0434)
3	−0.0655 (0.1837)	0.1058 (0.0236)	−0.2155 (0.0771)	0.0827 (0.0104)	−0.2246 (0.0269)	0.0053 (0.0460)
4	−0.0420 (0.1430)	0.0941 (0.0262)	−0.1318 (0.0672)	0.0752 (0.0122)	−0.1342 (0.0186)	0.0169 (0.0483)
5	−0.0822 (0.1474)	0.1131 (0.0267)	−0.2014 (0.0662)	0.0922 (0.0120)	−0.2070 (0.0217)	0.0270 (0.0435)
6	0.1179 (0.1607)	0.0688 (0.0196)	−0.0087 (0.0586)	0.0486 (0.0089)	−0.0047 (0.0146)	−0.0156 (0.0450)
7	−0.0503 (0.0829)	0.0880 (0.0242)	−0.0995 (0.0409)	0.0743 (0.0115)	−0.1010 (0.0111)	0.0374 (0.0371)
8	−0.0795 (0.0991)	0.0966 (0.0253)	−0.1443 (0.0473)	0.0813 (0.0119)	−0.1468 (0.0141)	0.0394 (0.0372)
9	−0.0813 (0.0907)	0.0954 (0.0258)	−0.1383 (0.0442)	0.0813 (0.0123)	−0.1408 (0.0130)	0.0432 (0.0382)

Table 17 Bias and estimated risk (in parentheses) of the M–H algorithm with $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayes					
	SEL		LINEX		ENTROPY	
	λ	θ	λ	θ	λ	θ
1	−0.7501 (0.9834)	0.1145 (0.0187)	−1.2196 (0.5599)	0.0962 (0.0082)	−1.0058 (0.0346)	0.0449 (0.0166)
2	−0.8009 (1.0855)	0.1224 (0.0200)	−1.2628 (0.5925)	0.1047 (0.0088)	−1.0553 (0.0382)	0.0564 (0.0159)
3	−0.8767 (1.1527)	0.1377 (0.0226)	−1.3434 (0.6410)	0.1207 (0.0099)	−1.1417 (0.0421)	0.0756 (0.0156)
4	−0.5544 (0.7147)	0.1168 (0.0212)	−0.9135 (0.3783)	0.1007 (0.0097)	−0.7337 (0.0216)	0.0591 (0.0200)
5	−0.7575 (0.8880)	0.1462 (0.0256)	−1.1322 (0.4924)	0.1306 (0.0115)	−0.9592 (0.0294)	0.0915 (0.0194)
6	−0.3128 (0.5196)	0.1070 (0.0192)	−0.7664 (0.2981)	0.0901 (0.0087)	−0.5355 (0.0150)	0.0455 (0.0197)
7	−0.3541 (0.4632)	0.1116 (0.0238)	−0.5908 (0.2273)	0.0989 (0.0114)	−0.4620 (0.0118)	0.0693 (0.0249)
8	−0.5193 (0.5588)	0.1298 (0.0253)	−0.7894 (0.2920)	0.1161 (0.0119)	−0.6497 (0.0157)	0.0843 (0.0233)
9	−0.4453 (0.4872)	0.1132 (0.0222)	−0.7040 (0.2561)	0.1001 (0.0105)	−0.5673 (0.0133)	0.0700 (0.0215)

Table 18 Bayesian confidence interval for $\lambda = 2$ and $\theta = 0.5$

Scheme	Bayesian confidence interval	
	λ	θ
1	(1.1056, 3.1061)	(0.2190, 0.9667)
2	(1.1002, 3.1434)	(0.2202, 0.9668)
3	(1.0457, 3.2321)	(0.2155, 0.9695)
4	(1.2348, 2.8898)	(0.2476, 0.9503)
5	(1.1365, 3.0676)	(0.2406, 0.9648)
6	(1.2723, 3.2512)	(0.2218, 0.9488)
7	(1.3942, 2.6143)	(0.2999, 0.9160)
8	(1.3087, 2.6982)	(0.2898, 0.9338)
9	(1.3325, 2.6418)	(0.3013, 0.9255)

Table 19 Coverage probability (CP) of Bayesian confidence interval for $\lambda = 2$ and $\theta = 0.5$

Scheme	CP	
	λ	θ
1	0.9850	0.9940
2	0.9810	0.9950
3	0.9770	0.9960
4	0.9670	0.9840
5	0.9770	0.9960
6	0.9940	0.9970
7	0.9690	0.9770
8	0.9680	0.9860
9	0.9670	0.9690

Table 20 Bayesian confidence interval for $\lambda = 5$ and $\theta = 0.6$

Scheme	Bayesian confidence interval	
	λ	θ
1	(2.5058, 6.6231)	(0.3212, 0.9862)
2	(2.4742, 6.5575)	(0.3330, 0.9868)
3	(2.4009, 6.5836)	(0.3473, 0.9884)
4	(2.9290, 6.4527)	(0.3541, 0.9835)
5	(2.7378, 6.4696)	(0.3729, 0.9883)
6	(3.0009, 7.0518)	(0.3393, 0.9829)
7	(3.4215, 6.1888)	(0.3930, 0.9688)
8	(3.2137, 6.2185)	(0.3930, 0.9796)
9	(3.2965, 6.2159)	(0.3984, 0.9755)

Table 21 Coverage probability (CP) of Bayesian confidence interval for $\lambda=5$ and $\theta = 0.6$

Scheme	CP	
	λ	θ
1	0.9630	1
2	0.9480	0.9990
3	0.9460	0.9990
4	0.9450	1
5	0.9470	1
6	0.9910	0.9980
7	0.9527	0.9776
8	0.9232	0.9875
9	0.9530	0.9780

Table 22 Estimated values of λ and θ

Scheme	NR method		SEM method	
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
1	0.09778	0.03752	0.04435	0.06966
2	0.06578	0.0185	0.04621	0.06537
3	0.07782	0.0105	0.06646	0.0182

Table 23 Estimated values of λ and θ

Scheme	Bayes estimates (MCMC method)					
	SEL		LINEX		ENTROPY	
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
1	0.0128	0.6612	0.0128	0.64958	0.0121	0.6273
2	0.0143	0.6244	0.0143	0.6140	0.01375	0.6140
3	0.0118	0.6560	0.0118	0.6444	0.01125	0.6223

Table 24 Estimated values of λ and θ

Scheme	Bayes estimates (Lindley's method)					
	SEL		LINEX		ENTROPY	
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
1	0.05119	0.05325	0.05240	0.050725	0.052409	0.04766
2	0.0455	0.07634	0.04420	0.06401	0.04419	0.048791
3	0.04022	0.06731	0.04368	0.0600	0.04370	0.05058

Table 25 Estimated values of λ and θ

Scheme	Bayes estimates (M–H method)					
	SEL		LINEX		ENTROPY	
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
1	0.0700	0.0337	0.0698	0.0329	0.0630	0.0094
2	0.0293	0.2374	0.0293	0.2241	0.0258	0.1319
3	0.0248	0.2912	0.0247	0.2742	0.0209	0.2742

Table 26 Confidence interval for λ and θ

Method	Scheme	Confidence interval	
		λ	θ
Bayesian	1	(0.0325, 0.1115)	(0.0019, 0.1515)
	2	(0.0128, 0.0515)	(0.0357, 0.6755)
	3	(0.0097, 0.04821)	(0.0468, 0.7504)
Asymptotic	1	(– 0.0208, 0.0938)	(0.10737, 0.1283)
	2	(0.01722, 0.05703)	(– 0.07901, 0.17856)
	3	(0.00252, 0.1031)	(0.03076, 0.06111)

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Appendix

Lindley Method

$$l_{\lambda\lambda} = \frac{\partial^2 l}{\partial \lambda^2} = -\frac{m}{\lambda^2} - 2\sum_{i=1}^{m} \frac{\theta(1-\theta)x_i^2 \mathrm{e}^{-\lambda x_i}}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2} - \sum_{i=1}^{m} \frac{\theta(1-\theta)R_i x_i^2 \mathrm{e}^{-\lambda x_i}}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2}.$$

$$l_{\theta\theta} = \frac{\partial^2 l}{\partial \theta^2} = -\frac{m}{\theta^2} + 2\sum_{i=1}^{m} \frac{\left[1-\mathrm{e}^{-\lambda x_i}\right]^2}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2} - \sum_{i=1}^{m} \frac{R_i\left[1-\mathrm{e}^{-\lambda x_i}\right]^2}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2}.$$

$$\begin{aligned} l_{\theta\lambda} = \frac{\partial^2 l}{\partial\theta\partial\lambda} = l_{\lambda\theta} = \frac{\partial^2 l}{\partial\lambda\partial\theta} \\ = -2\sum_{i=1}^{m}\left(\frac{x_i \mathrm{e}^{-\lambda x_i}}{\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta} + \frac{\left[1-\mathrm{e}^{-\lambda x_i}\right]x_i \mathrm{e}^{-\lambda x_i}(1-\theta)}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2}\right) \\ - \sum_{i=1}^{m}\left(\frac{R_i x_i \mathrm{e}^{-\lambda x_i}}{\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta} + \frac{R_i\left[1-\mathrm{e}^{-\lambda x_i}\right]x_i \mathrm{e}^{-\lambda x_i}(1-\theta)}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2}\right). \end{aligned}$$

$$
\begin{aligned}
l_{\lambda\lambda\lambda} = \frac{\partial^3 l}{\partial \lambda^3} = \frac{2m}{\lambda^3} \\
&+ 2\sum_{i=1}^{m}\left(\frac{x_i^3 \mathrm{e}^{-\lambda x_i}(1-\theta)}{\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta} - \frac{3x_i^3\left(\mathrm{e}^{-\lambda x_i}\right)^2(1-\theta)^2}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2} + \frac{2x_i^3\left(\mathrm{e}^{-\lambda x_i}\right)^3(1-\theta)^3}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3}\right) \\
&- \sum_{i=1}^{m}\left(\frac{2R_i\theta x_i^3\left(\mathrm{e}^{-\lambda x_i}\right)^2(1-\theta)^2}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3} - \frac{R_i\theta x_i^3 \mathrm{e}^{-\lambda x_i}(1-\theta)}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2}\right)
\end{aligned}
$$

$$
l_{\theta\theta\theta} = \frac{\partial^3 l}{\partial \theta^3} = \frac{2m}{\theta^3} - 2\sum_{i=1}^{m}\frac{2\left[1-\mathrm{e}^{-\lambda x_i}\right]^3}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3} - \sum_{i=1}^{m}\frac{2R_i\left[1-\mathrm{e}^{-\lambda x_i}\right]^3}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3}.
$$

$$
\begin{aligned}
l_{\lambda\theta\theta} = \frac{\partial^3 l}{\partial \lambda \partial \theta^2} = l_{\theta\theta\lambda} = \frac{\partial^3 l}{\partial \theta^2 \partial \lambda} \\
&= -2\sum_{i=1}^{m}\left(-\frac{2\left[1-\mathrm{e}^{-\lambda x_i}\right]x_i \mathrm{e}^{-\lambda x_i}}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2} - \frac{2\left[1-\mathrm{e}^{-\lambda x_i}\right]^2 x_i \mathrm{e}^{-\lambda x_i}(1-\theta)}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3}\right) \\
&- \sum_{i=1}^{m}\left(-\frac{2R_i\left[1-\mathrm{e}^{-\lambda x_i}\right]x_i \mathrm{e}^{-\lambda x_i}}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^2} - \frac{2R_i\left[1-\mathrm{e}^{-\lambda x_i}\right]^2 x_i \mathrm{e}^{-\lambda x_i}(1-\theta)}{\left[\mathrm{e}^{-\lambda x_i}(1-\theta)+\theta\right]^3}\right).
\end{aligned}
$$

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