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A MODIFIED TAYLOR COLLOCATION METHOD FOR PANTOGRAPH TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH HYBRID PROPORTIONAL AND VARIABLE DELAYS

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Abstract

In this work, high order pantograph type linear functional differential equations with hybrid proportional and variable delays is approximately solved by the modified Taylor matrix method. With this method these functional type differential equations are converted into the matrix form by the Taylor expansion method. The problems are reduced into a set of algebraic equations including Taylor coefficients. By determining the coefficients, the approximate solutions are calculated. Also, an error analysis technique with residual function is developed for the presented method. Some illustrative examples are given to demonstrate the efficiency and applicability of the method. The computer algebraic system Maple 15 is used for all calculations and graphs.

Keywords: Functional differential equations, Proportional and variable delays, Taylor polynomials and series, Numerical solutions, Residual error analysis.

KARIŞIK ORANLI VE DEĞİŞKEN GECİKMELİ PANTOGRAF TİPİ FONKSİYONEL DİFERANSİYEL DENKLEMLER İÇİN GELİŞTİRİLMİŞ TAYLOR SIRALAMA METODU

Özet

Bu çalışmada karışık oranlı ve değişken gecikmeli yüksek mertebe lineer pantograf tip fonksiyonel diferansiyel denklemler geliştirilmiş Taylor matris metotla yaklaşık olarak çözülmüştür. Bu metotla fonksiyonel tip diferansiyel denklemler Taylor açılım metodu ile matris forma dönüştürülür. Problemler Taylor katsayılı cebirsel denklem kümesine indirgenir. Katsayılar belirlenerek yaklaşık çözümler hesaplanır. Ayrıca, metot için kalan fonksiyonlu hata analizi geliştirilmiştir. Metodun verimlilik ve uygulanabilirliğini göstermek için bazı açıklayıcı örnekler verilmiştir. Tüm hesaplamalar ve grafikler için Maple 15 programlama dili kullanılmıştır.

Anahtar Kelimeler: Fonksiyonel diferansiyel denklemler, Nispi ve değişken gecikmeler, Taylor polinomları ve serileri, Nümerik çözümler, Residual hata analizi.

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1. Introduction

In this paper our aim is to obtain a numerical solution of pantograph type functional differential equations with mixed proportional and variable delays in the form

$$\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x)y^{(k)}(\alpha_{kj}x+\tau_{kj}(x)) = f(x) \qquad (1)$$

with the mixed conditions

$$\sum_{k=0}^{m-1}(a_{ik}y^{(k)}(\mathrm{a})+b_{ik}y^{(k)}(\mathrm{b})) = \mu_i \qquad (2)$$

where $i=0,1,2,\dots,m-1$.

Here $P_{kj}(x)$, $\tau_{kj}(x)$ and $f(x)$ are given and analytical functions on the interval $[a,b]$, α_{kj}, a_{ki}, b_{ki} and μ_i are given constants with $0<\alpha_{kj}<1$, $k=0,1,\dots,m$.

Functional differential equations with proportional and variable delays in the form (1) represent more general class of delay differential equations. They are frequently used to model a wide class of problems in many scientific fields as much as engineering, chemical reactions, mathematical physics, biology, ecology, economics, fluid and elastic mechanics, dynamical systems, population dynamics, signal processing and industrial processes. But most of these equations can not be solved exactly. Therefore, it is necessary to design efficient numerical methods to approximate their solutions.

Up to now, Dix [1] analyzed asymptotic behaviour of solutions of first order differential equations with variable delays, Liu et al. [2] established a new sufficient condition for existance, uniqeness of periodic solutions of a Liénard equation with delay, Schley et al.[3] and Zhang[7] studied about determination of linear stability properties for an ordinary differential equation with a varying time delay, Graef and Qian [4], Caraballo and Langa[5] examined attractivity of delay equations, Diblik et al. [6] gave a sufficient conditions for the existence of positive solutions of a scalar linear differential equation with time-dependent delay. Also, numerical methods are studied to solve these type of equations such as the rational approximate method [8], collocation method [9], multistep methods [10], Runge-Kutta methods [11,12] an one-leg-θ methods [13,14].

Here a new numerical technique is developed by modifying matrix methods which have been used by Sezer and coworkers [15-16]. Solutions obtained from this method is expressed in the form

$$y(x)=\sum_{N=0}^{N} y_n x^n. \qquad (3)$$

Here, $y_n, n=0,1,\dots,N$ are the Taylor coefficients which are needed to be computed.

2. Fundamental Matrix Relations

In this episode, equation (1) is transformed to a matrix equation. All relations which are needed are given respectively for this transformation. Therefore, let us first consider the approximate solution $y(x)$ and its derivatives $y^{(k)}(x), k=1,2,\dots,n$ defined by truncated Taylor series. Then we write the matrix form of (3) and its derivatives

$$y(x)=\mathbf{X}(x)\mathbf{Y} \qquad (4)$$

$$y^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k\mathbf{Y} \qquad (5)$$

where

$$\mathbf{X}(x)=\begin{bmatrix}1 & x & x^2 & \dots & x^N\end{bmatrix}$$

$$\mathbf{Y}=\begin{bmatrix}y_0 & y_1 & y_2 & \dots & y_N\end{bmatrix}^T$$

$$\mathbf{B}=\begin{bmatrix}0&1&0&0&\dots&0\\0&0&2&0&\dots&0\\0&0&0&3&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\dots&N\\0&0&0&0&\dots&0\end{bmatrix}$$

By putting $x\to\alpha_{kj}x+\tau_{kj}(x)$ in the relation (5), we obtain

$$y^{(k)}\left(\alpha_{kj}x+\tau_{kj}(x)\right)=\mathbf{X}\left(\alpha_{kj}x+\tau_{kj}(x)\right)\mathbf{B}^k\mathbf{Y}$$
$$=\mathbf{X}(x)\mathbf{M}(\alpha_{kj},\tau_{kj})\mathbf{B}^k\mathbf{Y} \qquad (6)$$

where

$$\mathbf{M}(\alpha_{kj},\tau_{kj}(x))=\begin{bmatrix}\binom{0}{0}(\alpha_{kj})^0(\tau_{kj}(x))^0 & \binom{1}{0}(\alpha_{kj})^0(\tau_{kj}(x))^1 & \binom{2}{0}(\alpha_{kj})^0(\tau_{kj}(x))^2 & \dots & \binom{N}{0}(\alpha_{kj})^0(\tau_{kj}(x))^N\\ 0 & \binom{1}{1}(\alpha_{kj})^1(\tau_{kj}(x))^0 & \binom{2}{1}(\alpha_{kj})^1(\tau_{kj}(x))^1 & \dots & \binom{N}{1}(\alpha_{kj})^1(\tau_{kj}(x))^{N-1}\\ 0 & 0 & \binom{2}{2}(\alpha_{kj})^2(\tau_{kj}(x))^0 & \dots & \binom{N}{2}(\alpha_{kj})^1(\tau_{kj}(x))^{N-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \binom{N}{N}(\alpha_{kj})^N(\tau_{kj}(x))^0\end{bmatrix}$$

If the relation (6) is substituted into equation (1) we gain the matrix equation as:

$$\sum_{k=0}^{m}\sum_{j=0}^{r}P_{kj}(x)\mathbf{X}(\mathrm{x})\mathbf{M}(\alpha_{kj},\tau_{kj}(x_r))\mathbf{B}^k\,\mathbf{Y}=f(x_r) \qquad (7)$$

2.1. Matrix Representation of Conditions

To find the matrix form of conditions, we substitude the relation (5) in equation (2). Hence, we get the following equation

$$\sum_{k=0}^{m-1}\left[a_{ik}\mathbf{X}(a)+b_{ik}\mathbf{X}(b)\right]\mathbf{B}^k\mathbf{Y}=\mu_i \qquad (8)$$

where i= $0,1,\dots,m-1$.

3. Method of Solution

Here by using the matrix equations (7) and (8), we get the approximate solution of (1) under the conditions (2). For this reason by placing the collocation points defined as

$$x_r=\mathrm{a}+\frac{b-a}{N}r,\qquad r=0,1,\dots,N \qquad (9)$$

into equation (7), we gain the system of matrix equations

for $r=0,\dots,N$,

$$\sum_{k=0}^{m}\sum_{j=0}^{r}P_{kj}(x_r)\mathbf{X}(x_r)\mathbf{M}(\alpha_{kj},\tau_{kj}(x_r))\mathbf{B}^k\,\mathbf{Y}=f(x_r) \qquad (10)$$

Thus, the compact form of the system (10) can be written as

$$\left(\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}\overline{\mathbf{X}}\overline{\mathbf{M}}_{kj}\mathbf{B}^{k}\right)\mathbf{Y}=\mathbf{F} \qquad (11)$$

where

$$\mathbf{P}_{kj}=diag\left[P_{kj}(x_0)\quad P_{kj}(x_1)\quad \dots\quad P_{kj}(x_N)\right]$$
$$\overline{\mathbf{X}}=diag\left[X(x_0)\quad X(x_1)\quad \dots\quad X(x_N)\right]$$
$$\overline{\mathbf{M}}_{kj}=diag\left[M_{kj}(\alpha_{kj},\tau_{kj}(x_0))\quad M_{kj}(\alpha_{kj},\tau_{kj}(x_1))\quad \dots\quad M_{kj}(\alpha_{kj},\tau_{kj}(x_N))\right]$$

$$\overline{\mathbf{B}}^{k}=\begin{bmatrix}\mathbf{B}^k\\ \mathbf{B}^k\\ \vdots\\ \mathbf{B}^k\end{bmatrix},\ \mathbf{Y}=\begin{bmatrix}y_0\\ y_1\\ \vdots\\ y_N\end{bmatrix},\ \mathbf{F}=\begin{bmatrix}f(x_0)\\ f(x_1)\\ \vdots\\ f(x_N)\end{bmatrix}.$$

In Equation (11), the full dimensions of the matrices $\mathbf{P}_{kj}, \overline{\mathbf{X}}, \overline{\mathbf{M}}_{kj}, \overline{\mathbf{B}}^{k}$, $\mathbf{Y}$ and $\mathbf{F}$, are respectively $(N+1)\times(N+1)$, $(N+1)\times(N+1)^2$, $(N+1)^2\times(N+1)^2$, $(N+1)^2\times(N+1)$, $(N+1)\times 1$ and $(N+1)\times 1$.
Also, the fundamental matrix equation (11) can be expressed in the form

$$\mathbf{WY}=\mathbf{F}\ \text{ or }\ [\,\mathbf{W};\mathbf{F}] \qquad (12)$$

where

$$\mathbf{W}=\left[w_{pq}\right]=\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}\overline{\mathbf{X}}\overline{\mathbf{M}}_{kj}\mathbf{B}^{k}$$

Similarly, by using relation (8), the matrix form of conditions is written in the form:

$$\mathbf{V}_i\mathbf{Y}=\mu_i\ \text{ or }\ [\,\mathbf{V}_i;\mu_i] \qquad (13)$$

where $i=0,1,\dots,m-1$

$$V_i=\sum_{k=0}^{m-1}\left[a_{ik}\mathbf{X}(a)+b_{ik}\mathbf{X}(b)\right]\mathbf{B}^k =\left[v_{i0}\quad v_{i1}\quad \cdots\quad v_{iN}\right].$$

Consequently, to obtain the solutions of Equation (1) under the conditions (2), we replace the row matrices (13) by any m rows of the matrix (12). So we get the following new augmented matrix

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]. \qquad (14)$$

If $rank(\widetilde{\mathbf{W}};\widetilde{\mathbf{F}})=rank(\widetilde{\mathbf{W}})=N+1$, then the unknown coefficients $y_0,y_1,\dots,y_N$ are uniquely determined from the system. Thus, if the determined coefficients are substituted into equation (3), the Taylor polynomial solution is obtained as the following

$$y_N(x)=\sum_{N=0}^{N} y_n x^n. \qquad (15)$$

Accuracy of the approximate solutions is checked by substituting the solutions into the equation (1)

$$E_N=\left|\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x){y_N}^{(k)}(\alpha_{kj}x+\tau_{kj}(x))-f(x)\right|. \quad (16)$$

Here if $y_N(x)\cong y(x)$, then $E_N(x)\cong 0$.

4. Residual Error Analysis

In this section, by using the residual correction method [17-19] we present an error estimation for the Taylor series solution of (1). Our aim is to predict the optimal M which is gived the minimal absolute error. By modifying the procedure [17-19] to Equation (1), we get the residual function for Taylor polynomial solution (3) as

$$R=\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x)y_N^{(k)}\left(\alpha_{kj}x+\tau_{kj}(x)\right).$$

If R is added into both sides of Equation (1), we get

$$\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x)e_N^{(k)}(\alpha_{kj}x+\tau_{kj}(x))=-R \qquad (17)$$

where

$$e_N(x)=y(x)-y_N(x).$$

If

$$\left\|e_N(x)-e_{N,M}\right\|\le\varepsilon$$

is sufficiently small where $e_{N,M}$ is Taylor series solution of (17), then the absolute error can be predicted bye $e_{N,M}$. Should $y_N(x)$ be Taylor series solution of (1), $y_{N,M}(x)=y_N(x)+e_{N,M}(x)$ named as corrected approximate solution is also a solution of (1).

5. Illustrative Problems

In this section, some problems are given to understand how the method is progressing and how effective and precise it is.

Problem 1. Let us consider the following functional differential equation with the initial conditions

$$y''(x)+y'(x-sinx)+2xy(x)=\mathrm{f}(x)$$
$$y(0)=-1,\qquad y'(0)=0$$

where $f(x)=2-2\mathrm{xcosx}-2\mathrm{sinx}+2\mathrm{sinxcosx}+2x^3$
For the interval [0,1] the set of collocation points for $N=2$ becomes $\left\{x_0=0,x_1=\frac{1}{2},x_2=1\right\}$. By using the mentioned procedure and solving the system of

corresponding augmented matrix $[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]$ the Taylor coefficients are uniquely determined as $Y=[-1 \quad 0 \quad 1]$. Finally, if these determined coefficients are substituted into (3), the approximate solution is obtained as $y(x)=x^2-1$. That is also the exact solution of the equation.

Problem 2. For the present example, we examine the equation as

$$2y'(x)-xy(x)+xe^{2x^2}y(x-x^2)=4e^{2x}$$

under the initial condition $y(0)=1$ which has the exact solution as $y(x)=e^{2x}$. By using the way in section 3 and taking $N=5,7,10,14$ we compute the approximate solutions. For $N=5$ the numerical solution which is obtained as

$$y_5(x)=1+2.0000x+1.97936x^2+1.47227x^3 +0.323343x^4+0.610519x^5.$$

All other results are given with tables and figure. Numerical datas of solutions for $(N,M)=(7,10),(10,13)$ are presented in Table 1. According to this table, it is obvious that as the values of *N,M* are increased the approximate solutions $y_N(x)$ and $y_{N,M}(x)$ are approaches to the exact solution. In Table 2, it is seen that the errors are decrease when *N,M* values get bigger. Additionaly, the results are shown that $e_{N,M}(x)$ calculated by the residual function is closer to zero than $e_N(x)$. Also, in Figure 1 the graphics of corrected absolute errors are plotted for different *(N,M)*.
Figure 1 shows that if the values *N,M* increase then the absolute errors decrease.

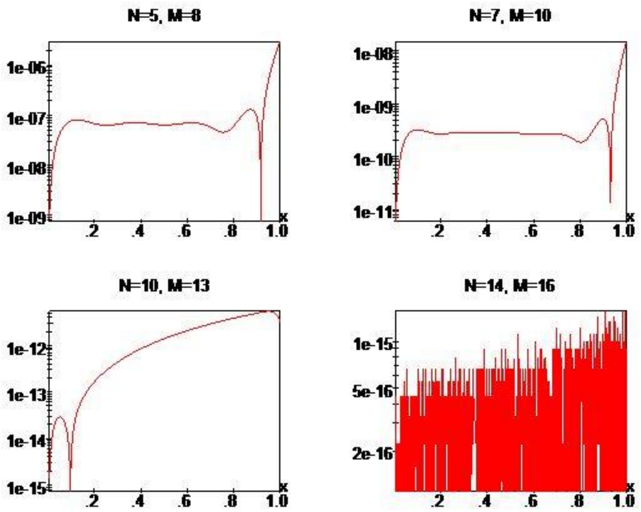

Figure 1."Corrected absolute errors"of Problem 2 for varied *N,M* values.

Problem 3. For the third example we choose the following problem as

$$y'''(x)-y''(x-x^2)+y(x)=x-e^{x^2-x}$$

with the initial conditions

$$y(0)=1, y'(0)=0, y''(0)=1.$$

The exact solution of this problem is $y(x)=x+e^{-x}$.
Outputs of this equation are given in Table 3, Table 4
In Figure 2, corrected absolute errors are presented for different *N,M*.

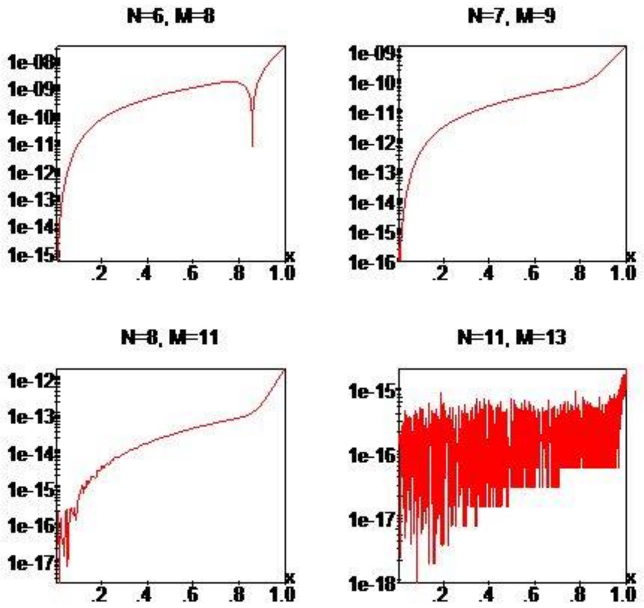

Figure 2."Corrected absolute errors"of Problem 3 for varied *N,M* values.

6. Conclusion

In this paper, we are presented a matrix method based on the Taylor polynomial to solve the pantograph type functional differential equations with hybrid proportional and variable delays. The residual error function is defined for these type of equations to predict the absolute error. Also, it is indicated how the mentioned method and the error analysis prosedures are perfomed on some problems When the problems are examined it is seen that the Taylor polynomial coefficients are found very easily by using computer program written in Maple 15. The numerical results show that if truncation limit N is increased, it can be seen that approximate solutions get closer to the exact solutions. In addition, the technique can also be extended to other type of equations and systems with some modifications.

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Table 1. Comparisons of solutions of Problem 2 for $(N, M) = (7.10), (10{,}13)$.

x_i	"Exact solution" $y(x_i)= e^{2x_i}$	"Approximate" "solution"for N=7 $y_7(x_i)$	"Corrected" "approximate" solution" $y_{7,10}(x_i)$	"Approximate" "solution"for N=10 $y_{10}(x_i)$	"Corrected" "approximate" "solution" $y_{10,13}(x_i)$
			Numerical values		
0	1	1.00000000000	1.00000000000	1.00000000000	1
0.2	1.49182469764	1.49182370972	1.49182469793	1.49182469792	1.49182469766
0.4	2.22554092849	2.22553996766	2.22554092879	2.22554092876	2.22554092851
0.6	3.32011692274	3.32011612513	3.32011692309	3.32011692299	3.32011692276
0.8	4.95303242440	4.95303209614	4.95303242475	4.95303242459	4.95303242437
1.0	7.38905609893	7.38902007967	7.38905608402	7.38905608372	7.38905609884

Table 2. Comparisons of the absolute errors of Problem 2 for $(N, M) = (10{,}13), (14{,}16)$.

x_i	"Actual absolute errors"for N=10 $e_{10}(x_i)$	"Estimated absolute errors"for N=10 and M=13 $e_{10,13}(x_i)$	"Corrected absolute errors" for N=10 and M=13 $E_{10,13}(x_i)$
0	0	0	0
0.2	0.2712e-009	0.2697e-009	0.3249e-011
0.4	0.2795e-009	0.2763e-009	0.7633e-011
0.6	0.2636e-009	0.2645e-009	0.1215e-011
0.8	0.1865e-009	0.1887e-009	0.2545e-010
1.0	0.1521e-007	0.1521e-007	0.8974e-010
x_i	"Actual absolute errors" for N=14 $e_{14}(x_i)$	"Estimated absolute errors" for N=14 and M=16 $e_{14,16}(x_i)$	"Corrected absolute errors" for N=14 and M=16 $E_{14,16}(x_i)$
0	0	0	0
0.2	0.1112e-011	0.1957e-014	0.3512e-017
0.4	0.2059e-011	0.1934e-014	0.3475e-017
0.6	0.3849e-011	0.1864e-014	0.3352e-017
0.8	0.5783e-011	0.1640e-014	0.2942e-017
1.0	0.7561e-012	0.1461e-012	0.3007e-015

Table 3 Comparisons of exact and numerical solutions of Problem 3 for (N,M)=(6,8),(8,11)

x_i	"Exact solution" $y(x_i)= x_i + e^{-x_i}$	"Approximate" "solution"for N=6 $y_6(x_i)$	"Corrected" "approximate" solution for N=6 and M=8 $y_{6,8}(x_i)$	"Approximate solution"for N=8 $y_8(x_i)$	"Corrected approximate solution"for N=6 and M=8 $y_{8,11}(x_i)$
			Numerical values		
0	1	1.00000000000	1.000000000000	1.00000000000	1.000000000
0.2	1.01873075307	1.01873072799	1.018730753016	1.0187307530167	1.01873075307
0.4	1.07032004604	1.070319897493	1.070320045712	1.0703200457125	1.07032004603
0.6	1.14881163609	1.148811240247	1.148811635272	1.1488116352727	1.14881163609
0.8	1.24932896412	1.249328972003	1.249328962894	1.2493289628955	1.24932896410
1.0	1.36787944117	1.367889625483	1.367879467102	1.3678794671047	1.36787944113

Table 4. Comparison of the absolute errors of Problem 3 for $(N,M)=(8,11),(11,13)$.

x_i	"Actual absolute errors" for N=8 $e_8(x_i)$	"Estimated absolute errors" for N=8 and M=11 $e_{8,11}(x_i)$	"Corrected absolute errors" for N=8 and M=11 $E_{8,11}(x_i)$
0	0	0	0
0.2	0.61338e-010	0.61335e-010	0.27104e-012
0.4	0.32316e-009	0.32314e-009	0.21694e-011
0.6	0.82131e-009	0.82127e-009	0.73384e-011
0.8	0.12217e-008	0.12216e-008	0.17422e-010
1.0	0.25933e-007	0.25934e-007	0.35556e-010
x_i	"Actual absolute errors" for N=11 $e_{11}(x_i)$	"Estimated absolute errors" for N=11 and M=13 $e_{11,13}(x_i)$	"Corrected absolute errors" for N=11 and M=13 $E_{11,13}(x_i)$
0	0	0	0
0.2	0.30758e-014	0.30731e-014	0.26779e-017
0.4	0.15231e-013	0.15218e-013	0.12973e-016
0.6	0.38220e-013	0.38188e-013	0.32374e-016
0.8	0.73922e-013	0.73861e-013	0.62131e-016
1.0	0.16641e-011	0.16626e-011	0.15089e-014