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Operational Matrix by Hermite Polynomials for Solving Nonlinear Riccati Differential Equations

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Abstract

This paper studies the potential to ensure a numerical solution of nonlinear Riccati differential equations with an effective method, namely operational matrix which is derived by Hermite polynomials with the sense of Caputo derivative. In order to solve the Riccati differential equations, the complete problem is simplified with the operational matrix obtained. To achieve this goal, the proposed approach converts the fractional differential equations (FDEs) into a set of algebraic equations. We then construct a matrix with the algebraic equations and extra equations extracted from initial conditions. Therefore, we achieve the solution by solving these algebraic equations given in a matrix sense. In order to show the efficiency of the proposed idea, we show a number of illustrative examples in which the results confirm the applicability of the suggested approach.

Key words and phrases: Fractional differential equations, Caputo, Operational matrix, Hermite, Riccati.
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1 Introduction

In recent years, fractional calculus with its wide range of applications has been gaining a significant interest in many branches of engineering and applied science [1]. The main motivation is to realistically model such a real-world problem through a set of efficient mathematical tools. Fractional differential equations (FDEs) have been a popular way of defining real-world applications. Therefore, a lot of previous numerical studies have attempted to provide the solutions of FDEs with initial conditions. Typical examples of numerical techniques for solving FDEs are eigenvector method [2], Adomian decomposition method [3], power series method [4], collacation method [5] and operational matrix method [6, 7, 8, 9]. The operational matrix method is one of most popular method to solve different kinds of linear and nonlinear FDEs [10]. Its underlying feature is to simplify the whole problem by generating a number of algebraic equations. Depending on the number of initial conditions, a certain number of algebraic equations are extracted from the initial conditions. The complete problem therefore requires the solution of these algebraic equations in a simple way. To accomplish this, orthogonal polynomials have recently been utilized to derive the operational matrices of fractional derivatives [11, 12, 13, 14, 15, 16]. A critical effort was previously placed on the solution of the Riccati differential equation [17]. Due to the high volume of application areas, the Riccati differential equation is of paramount significance. The following Riccati differential equation is dealt with

$$f_1(x)D^a u(x) + f_2(x)u^2(x) + f_3(x)u(x) = g(x), \quad x \in (0,1), \quad 0 < \alpha \leq 1 \tag{1.1}$$

subject to the initial conditions up to n

$$u^{(1)}(0) = d_1, \quad k = 0, 1, 2, ., n-1, \tag{1.2}$$

where α indicates the order of the fractional derivative, d_k is a constant, $f_1(x)$, $f_2(x)$ and $f_3(x)$ are certain functions with a condition that $f_1(x) \neq 0$ and $g(x)$ is a given source function.

A number of previous studies based on the operational matrix method with a specific orthogonal polynomial have been achieved by approximate solutions for Riccati differential equation [18, 19, 20]. This paper presents an efficient numerical solution to solve nonlinear Riccati differential equation via deriving the operational matrix by Hermite polynomials in the sense of Caputo derivative. We first define the Riccati equation in integral form with the obtained operational matrix, resulting in a number of algebraic equations. We then extract extra algebraic equations from the initial conditions. An algebraic system is constructed with these equations to be solved to obtain approximate solutions. Therefore, the whole problem will eventually necessitate to solve a system of algebraic equations which simplifies the problem greatly. The main advantage of the proposed strategy is its low complexity structure which requires a small number of iterations to obtain good results. This also makes the proposed strategy more practical with a high speed of solution. Another advantage is the scope of the test examples including exact solutions in complex form, instead of only simple form of polynomials. The performance of the proposed idea is tested through a number of Riccati differential equations. The performance outputs prove the efficiency of the proposed method. The following sections present the details of the paper.

2 Method of Solution

2.1 Hermite Polynomials

Hermite polynomials are defined on $(-\infty, \infty)$ with this analytical formula: [21]

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k (2x)^{i-2k}}{k!(i-2k)!}, \tag{2.3}$$

where $\lfloor \frac{i}{2} \rfloor$ denotes the smallest natural number greater than $\frac{i}{2}$. Hermite polynomials are orthogonal polynomials [22]

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) = h_j \delta_{ij} \tag{2.4}$$

where $h_j = 2^j j! \sqrt{\pi}$ and δ_{ij} is the Kronecker function.

2.2 Hermite Operational Matrix of Caputo Derivative

In this section, our target is the derivation of an operational matrix for Hermite polynomials. Let $u(x) \in L^2(\Omega)$-$(\Omega = (-\infty, \infty))$. Then $u(x)$ can be defined in association with Hermite polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j H_j(x) \tag{2.5}$$

Then, the coefficient a_j is written as

$$a_j = \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} H_j(x) u(x) w(x) dx \qquad j = 0, 1, 2.... \tag{2.6}$$

where $w(x) = e^{-x^2}$ is the weight function of Hermite polynomials. The first $N+1$ terms of Hermite polynomials appear in

$$u_N(x) = \sum_{j=0}^{N} a_j H_j(x) = A^T \phi(x) \tag{2.7}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}$$

and

$$\phi(x) = \begin{bmatrix} \phi_0 & \phi_1 & \dots & \phi_N \end{bmatrix}.$$

Theorem 2.1. *Let $\phi(x)$ be the Hermite vector and $v > 0$. Then*

$$D^v(x) \simeq \mathbf{D}^{(v)} \phi(x) \tag{2.8}$$

where $\mathbf{D}^{(v)}$ indicates the $(N+1) \times (N+1)$ operational matrix of fractional derivative of order v in the Caputo sense that can be given as

$$\mathbf{D}^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \Omega_v(i,0) & \Omega_v(i,1) & \Omega_v(i,2) & \dots & \Omega_v(i,N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Omega_v(N,0) & \Omega_v(N,1) & \Omega_v(N,2) & \dots & \Omega_v(N,N) \end{pmatrix},$$

where the element of operational matrix $D^{(v)}$ can be found by

$$\Omega_v(i,j) = \sum_{i=0}^{\lfloor \frac{n-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \frac{(-1)^{(i+r)} 2^{(n-2i+j-2r)} n! j! \frac{\Gamma(n-2i+j-2r+1)}{2}}{(j-2r)! i! r! \Gamma(n-2i+1-v)}, j = 0, 1, ..., N. \tag{2.9}$$

Proof. We apply Caputo derivative to the analytic form of Hermite polynomials as:

$$D^v H_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i 2^{n-2i} D^v(x^{n-2i})}{i!(n-2i)!} \tag{2.10}$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i 2^{n-2i} (x^{n-2i-v})}{i!} \tag{2.11}$$

Approximating x^{n-2i-v} by the $N+1$ Hermite polynomials, we get

$$x^{(n-2i-v)} = \sum_{j=0}^{N} c_j H_j(x) \tag{2.12}$$

where c_j is given by Eq. (2.6) as

$$c_j = \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} H_j(x) u(x) w(x) dx \qquad j = 0, 1, 2.... \tag{2.13}$$

Then, in the light of Eq. (2.12) and Eq. (2.13), we obtain

$$D^v H_N(x) = \sum_{j=0}^{N} \Omega(i,j) H_j(x) \tag{2.14}$$

where

$$\Omega_v(i,j) = \sum_{i=0}^{\lfloor \frac{n-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^{(i+r)} 2^{(n-2i+j-2r)} n! j! \frac{\Gamma(n-2i+j-2r+1)}{2}}{(j-2r)! i! r! \Gamma(n-2i+1-v)}, j = 0, 1, ..., N \tag{2.15}$$

2.3 Operational Matrix Method for Nonlinear FDEs

This section applies the Hermite operational matrix to the nonlinear FDEs solution. The nonlinear FDE is given

$$D^v u(x) = F(x, D^{\beta_1} u(x), D^{\beta_2} u(x), ..., D^{\beta_k} u(x)) \tag{2.16}$$

subject to the following initial conditions

$$u^{(i)}(0) = d_i, \qquad i = 0, 1, ..., m-1. \tag{2.17}$$

Here F is not linear with the conditions of $m-1 < v \leq m$, and $0 < \beta_1 < \beta_2 < \ < \beta_k < v$. Also, $d_i, i = 0, 1, 2, , m-1$ are initial conditions. The proposed approximate solution is

$$u_N(x) = \sum_{j=0}^{N} c_j H_j(x) = C^T \phi(x) \tag{2.18}$$

where the coefficients c_i are to be determined in a final step. $D^v u(x)$ and $D^{(\beta_1)} u(x)$ are given in matrix form as

$$D^v u(x) \simeq C^T D^v \phi(x) = C^T D^{(v)} \phi(x) \tag{2.19}$$

$$D^{\beta_k} u(x) \simeq C^T D^{\beta_k} \phi(x) = C^T D^{(\beta_j)} \phi(x). \tag{2.20}$$

Substituting equations (2.19) - (2.21) into (2.17), we obtain

$$D^v u(x) = F(C^T \phi(x), C^T D^{\beta_1} \phi(x), ..., C^T D^{\beta_k} \phi(x)), \tag{2.21}$$

where F is the nonlinear function given as $F = g(x) - f_2(x)u^2(x) - f_3(x)u(x)$ and $u(x)$ is the solution function. Considering Eqs. (2.18)-(2.20), the nonlinear fractional differential can be solved in matrix form Eq. (2.21). Then, $(N-m+1)$ roots of the Hermite polynomials are used for solving the nonlinear algebraic equation system. In other words, we collocate the equation system with $(N-m+1)$ roots and (m) equations from the conditions. Therefore, the equation system is solved with Newton iteration method and finally the unknown

$$C^T = \left[\begin{array}{cccc} c_0 & c_1 & \dots & c_N \end{array} \right]$$

coefficients are found. Then, the approximate solution form given in (2.19) will be found by the coefficients.

3 Numerical Examples

Example 3.1. *We first consider the following nonlinear Riccati differential equation*

$$D^\alpha u(x) + u^2(x) = 1, \tag{3.22}$$

subject to initial condition

$$u(0) = 0. \tag{3.23}$$

For $\alpha = 1$, the exact solution of this equation is given as

$$u(x) = \frac{e^{2x-1}}{e^{2x+1}} \tag{3.24}$$

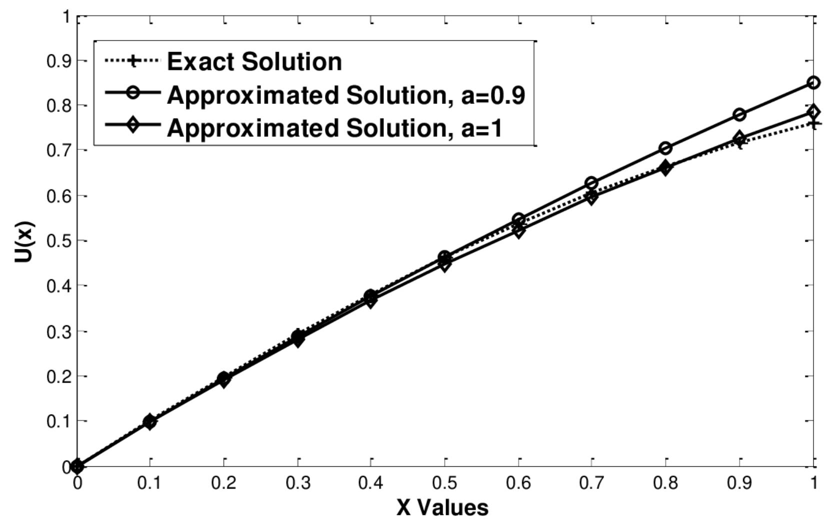

Figure 1: The results of the exact and approximate solutions, for $\alpha = 0.9$ and $\alpha = 1$.

For $N = 2$, we obtain the following derivative

$$\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}.$$

After applying the technique described in the previous section, we obtain

$$C^T = \begin{bmatrix} -0.09 & 0.5 & -0.045 \end{bmatrix}.$$

The proposed method yields the approximate solution as

$$u(x) = C^T\Phi(x) = x - 0.18x^2 \tag{3.25}$$

We then plot the results of the exact and approximate solutions as illustrated in Figure 1. The results show a highly-accurate approximation with the exact solutions by converging the α value to 1.

Example 3.2. *We consider the following nonlinear Riccati differential equation*

$$D^\alpha u(x) + u^2(x) - 2u(x) = 1, \qquad 0 < x, \alpha \leq 1 \tag{3.26}$$

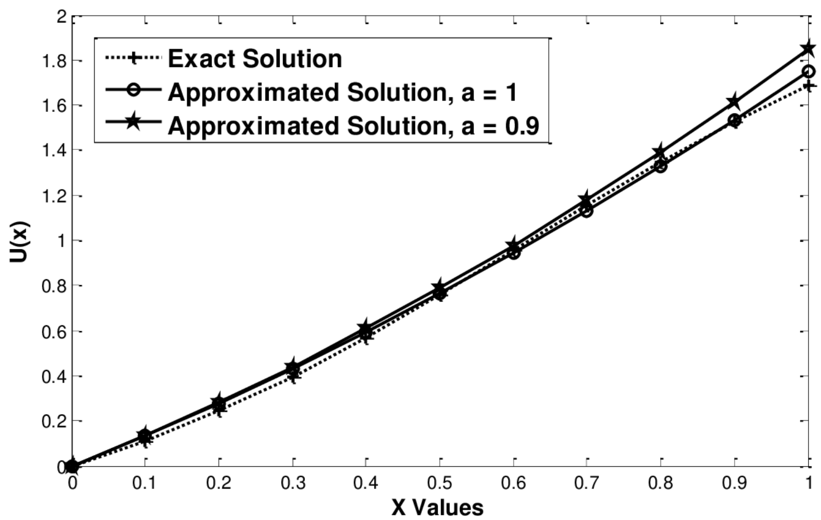

Figure 2: The results of the exact and approximate solution, for $\alpha = 0.3$ and $\alpha = 1$.

subject to the initial condition

$$u(0) = 0. \tag{3.27}$$

For $\alpha = 1$, the exact solution of this equation is given as

$$u(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}) \tag{3.28}$$

After applying the technique described in previous section, we obtain

$$C^T = \left[\begin{array}{ccc} 0.225 & 0.65 & 0.1125 \end{array}\right]$$

The proposed method obtains the approximate solution as:

$$u(x) = 1.3x + 0.45x^2. \tag{3.29}$$

Similarly, Fig. 2 presents the results of exact and approximate solutions. The results clearly show a close match with the exact solutions.

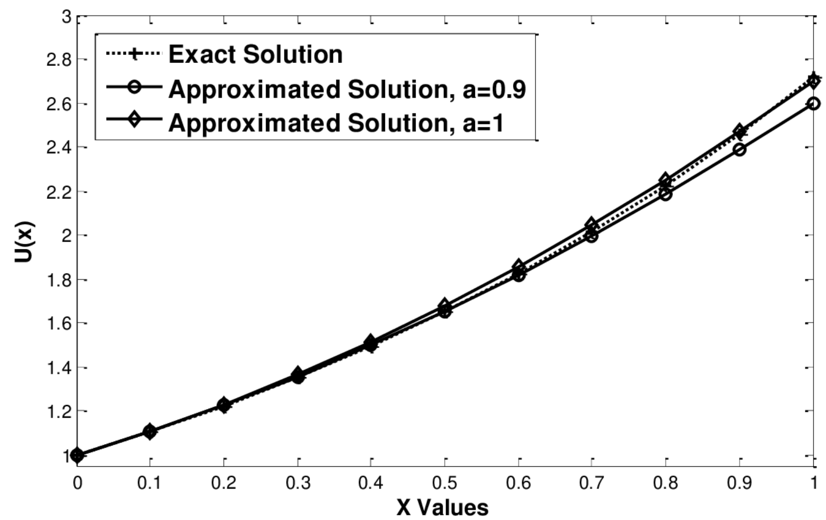

Figure 3: The results of the exact and approximate solutions, for $\alpha = 0.9$ and $\alpha = 1$.

Example 3.3. *The final nonlinear Riccati differential equation is*

$$D^{\alpha}u(x) - u^2(x) + e^x u(x) = e^x \tag{3.30}$$

subject to the initial conditions

$$u(0) = 1. \tag{3.31}$$

For $\alpha = 1$, the exact solution of this equation is given as $u(x) = e^x$. After applying the technique described in previous section, we obtain

$$C^T = \begin{bmatrix} 1.3 & 0.5 & 0.15 \end{bmatrix}.$$

The proposed method obtains the approximate solution as

$$u(x) = 1 + x + 0.6x^2 \tag{3.32}$$

Fig. 3 indicates the results of the exact and approximate solutions with a good agreement.

4 Conclusion

This paper introduced the derivation of the operational matrix by Hermite polynomials to solve the nonlinear Riccati differential equations. The proposed method can be applied to different type of nonlinear fractional differential equations (FDEs). The main idea is to simplify the problem by converting the FDEs into a group of algebraic equations with given initial conditions. As a result, by solving the algebraic equations, we achieve either the exact or approximate solutions. It is shown that the method presents a good level of approximation with accurate results.

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