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C_m-SUPERMAGIC LABELING OF FRIENDSHIP GRAPHS

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Abstract. The friendship graph F_n^m is obtained by joining n copies of the cycle graph C_m with a common vertex. In this work, we investigate the C_m-supermagic labeling of friendship graphs.

Keywords: Magic labeling, covering, friendship graphs.

AMS Subject Classification: 05C78.

1. Introduction and Preliminaries

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling was first introduced by Rosa [6] in 1966. Since then there are various types of labeling that have been studied and developed (see [1]).

A finite simple graph $G(V, E)$ admits an H-covering if every edge of G belongs to a subgraph of G isomorphic to H. Guitérrez and Lladó [2] introduced the notion of an H-magic labeling as follows. Let $G = (V, E)$ be a finite simple graph that admits H-covering. A bijection function $\lambda : V \cup G \to \{1, 2, 3, \ldots, |V| + |E|\}$ is called H-magic labeling of G if for every subgraph $H' = (V', E')$ of G isomorphic to H, $\sum\limits_{v \in V'} \lambda(v) + \sum\limits_{e \in E'} \lambda(e) = m(\lambda)$ is constant. Here $m(\lambda)$ is called as magic sum. The graph G is called H-supermagic if $\lambda(V) = \{1, 2, 3, \ldots, |V|\}$.

Many researches have studied H-supermagic labeling. For example: In [5] Maryati, Baskoro and Salaman studied path-supermagic labeling. Roswitha et al. [7] investigated H-supermagicness of some classes of graphs such as a Jahangir graph, a wheel graph for even n, and a complete bipartie graph $K_{m,n}$ for $m = 2$. C_4-supermagic labelings of the cartesian product of paths and graphs was given by Kojima [3]. Selvagopal and Jeyanthi [8] showed that polygonal snake graphs has C_m-supermagic labeling.

The friendship graph F_n^m is obtained by joining n copies of the cycle graph C_m with a common vertex. Different kind of labelings of friendship graphs have been investigated:

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Shalini and Kumar [9] investigated friendship graphs with four types of labeling such that Harmonious, Cordial, distance antimagic labeling and sum labeling. Prime labeling of friendship graphs given by Meena and Vaithilingam [10]. Edge vertex prime labeling of friendship graphs studied by Parmar [11]. Harmonious labeling of certain graph including friendship graphs investigated by Tanna [12]. In [13], Prasanna and Suhakar gave algorithms to enumerate all non-isomorphic Vertex and Edge Magic Total Labeling on cycle graphs, wheels, Fan Graphs and Friendship graphs. Radhika1 and Selvi [14] showed that Friendship graph F_2^3 is θ- graceful. Daoud and A.N. Elsawy [15] proved that double fan graphs, quadrilateral friendship graphs, and butterfly graphs are edge even graceful. Llado and Moragas [4] studied some C_m-supermagic graphs including friendship graphs. In this work, we present different kind of C_m-supermagicness of friendship graphs.

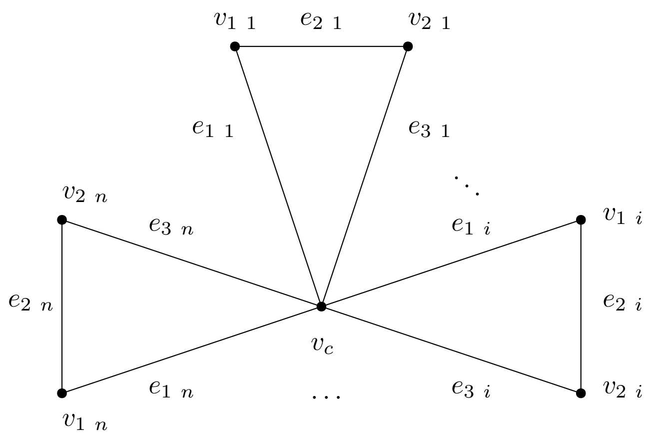
2. Results

Theorem 2.1. *The Friendship graph $F_n^3; n \geq 2$, admits a C_3-supermagic labeling.*

Proof. F_n^3 has $2n+1$ vertices and $3n$ edges. The vertices and edges of F_n^3 are denoted as follows:

$$\begin{aligned} &V = \{v_c\} \cup \{v_{j\ i} : j = 1, 2,\ i = 1, \dots, n\} \\ &E = \{e_{1\ i} : e_{1\ i} = v_c v_{1\ i} : i = 1, \dots, n\} \cup \{e_{2\ i} : e_{2\ i} = v_{1\ i} v_{2\ i} : i = 1, \dots, n\} \\ &\quad \cup \{e_{3\ i} : e_{3\ i} = v_{2\ i} v_c : i = 1, \dots, n\} \end{aligned}$$

where v_c is the common vertex.



To define a bijection $\lambda : V \cup E \to \{1, 2, 3, \dots, |V| + |E|\}$, we need to investigate two cases.
Case1: n is odd:

$$\begin{aligned} \lambda(v_c) &= 1, \\ \lambda(v_{1\ i}) &= 1 + i,\ i = 1, 2, 3, ..., n, \\ \lambda(v_{2\ i}) &= 2n + 2 - i,\ i = 1, 2, 3, ..., n, \\ \lambda(e_{1\ i}) &= 2n + 1 + i,\ i = 1, 2, 3, ..., n, \\ \lambda(e_{2\ i}) &= \begin{cases} 3n + \frac{n+1}{2} + i & , i = 1, 2, 3, ..., \frac{n+1}{2} \\ 2n + \frac{n+1}{2} + i & , i = \frac{n+1}{2} + 1, \dots, n \end{cases} \\ \lambda(e_{3\ i}) &= \begin{cases} 5n + 3 - 2i & , i = 1, 2, 3, ..., \frac{n+1}{2} \\ 6n + 3 - 2i & , i = \frac{n+1}{2} + 1, \dots, n \end{cases} \end{aligned}$$

Here, for all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, 2n+1\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_3, we have

$$\sum_{v\in V'} \lambda(v) = \lambda(v_c) + \lambda(v_{1\ i}) + \lambda(v_{2\ i}) = 2n + 4$$
$$\sum_{e\in E'} \lambda(e) = \lambda(e_{1\ i}) + \lambda(e_{2\ i}) + \lambda(e_{3\ i}) = 10n + \frac{n+1}{2} + 4$$
$$m(\lambda) = \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 12n + \frac{n+1}{2} + 8.$$

Case2: n is even:

$$\lambda(v_c) = n + 1 + \frac{n}{2},$$
$$\lambda(v_{1\ i}) = i,\ i = 1, 2, 3, ..., n,$$
$$\lambda(v_{2\ i}) = \begin{cases} 2n + 2 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ n + 1 + \frac{n}{2} - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$
$$\lambda(e_{1\ i}) = \begin{cases} 2n + 2 + \frac{n}{2} - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ 3n + 2 - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$
$$\lambda(e_{2\ i}) = 3n + 1 + i,\ i = 1, 2, 3, ..., n,$$
$$\lambda(e_{3\ i}) = 5n + 2 - i,\ i = 1, 2, 3, ..., n,$$

Similarly, for all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, 2n+1\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_3, we have

$$\sum_{v\in V'} \lambda(v) = \lambda(v_c) + \lambda(v_{1\ i}) + \lambda(v_{2\ i}) = \begin{cases} 3n + \frac{5}{2} + \frac{n}{2} + \frac{i}{2} & , i = 1, 3, 5, ..., n-1 \\ 3n + 2 + \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$
$$\sum_{e\in E'} \lambda(e) = \lambda(e_{1\ i}) + \lambda(e_{2\ i}) + \lambda(e_{3\ i}) = \begin{cases} 10n + \frac{9}{2} + \frac{n}{2} - \frac{i}{2} & , i = 1, 3, 5, ..., n-1 \\ 11n + 5 - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$
$$m(\lambda) = \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 14n + 7.$$

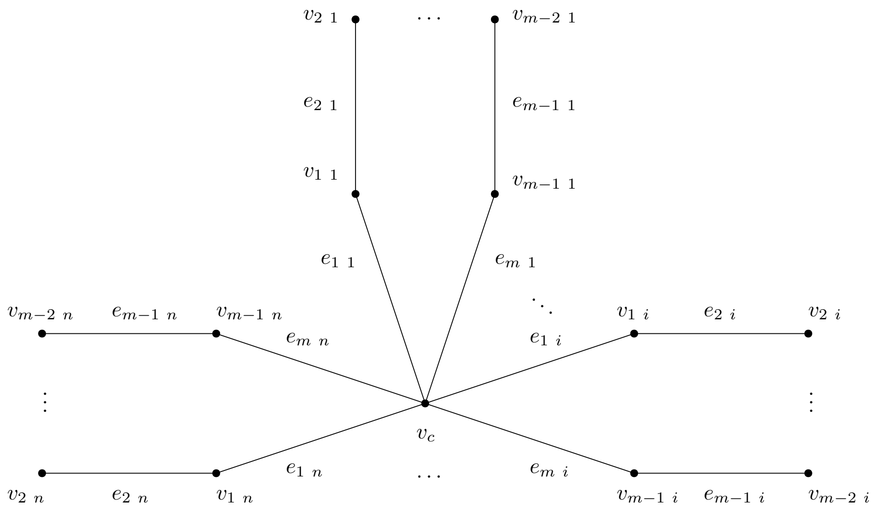
Hence F_n^3 admits a C_3-supermagic labeling. □

Theorem 2.2. *The Friendship graph of C_m, F_n^m, admits a C_m-supermagic labeling.*

Proof. The F_n^m has $(m-1)n + 1$ vertices and mn edges. The vertices and edges of F_n^m are denoted as follows:

$V = \{v_c\} \cup \{v_{j\ i} : j = 1, 2, 3, \ldots, m-1\ i = 1, \ldots, n\}$
$E = \{e_{1\ i} : e_{1\ i} = v_c v_{1\ i} : i = 1, \ldots, n\} \cup \{e_{j\ i} : e_{j\ i} = v_{j-1\ i} v_{j\ i} : j = 2, 3, 4, \ldots, m-1\ i = 1, \ldots, n\}$
$\cup \{e_{m\ i} : e_{m\ i} = v_{m-1\ i} v_c : i = 1, \ldots, n\}$

where v_c is the common vertex.



To define a bijection $\lambda : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$, we need to investigate 4 cases.
Case1: m is even and n is odd:

$$\lambda(v_c) = 1,$$

$$\lambda(v_{1\ i}) = 1 + i,\ i = 1, 2, 3, ..., n,$$

$$\lambda(v_{2\ i}) = \begin{cases} n + \frac{n+1}{2} + i & , i = 1, 2, 3, ..., \frac{n+1}{2} \\ \frac{n+1}{2} + i & , i = \frac{n+1}{2} + 1, \ldots, n \end{cases}$$

$$\lambda(v_{3\ i}) = \begin{cases} 3n + 3 - 2i & , i = 1, 2, 3, ..., \frac{n+1}{2} \\ 4n + 3 - 2i & , i = \frac{n+1}{2} + 1, \ldots, n \end{cases}$$

$$\lambda(v_{j\ i}) = \begin{cases} 1 + (j-1)\, n + i & , j = 4, 6, 8, ..., m-2,\ i = 1, 2, 3, ..., n \\ 2 + jn - i & , j = 5, 7, 9, ..., m-1,\ i = 1, 2, 3, ..., n \end{cases}$$

$$\lambda(e_{j\ i}) = \begin{cases} (m-1)\, n + 1 + (j-1)\, n + i & , j = 1, 3, 5, ..., m-1,\ i = 1, 2, 3, ..., n \\ (m-1)\, n + 2 + jn - i & , j = 2, 4, 6, ..., m,\ i = 1, 2, 3, ..., n \end{cases}$$

Here, for all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, (m-1)\, n + 1\}$ and for any subgraph

$H' = (V', E')$ isomorphic to C_m, we have

$$\begin{aligned}
\sum_{v\in V'} \lambda(v) &= \lambda(v_c) + \lambda(v_{1\ i}) + \lambda(v_{2\ i}) + \lambda(v_{3\ i}) + \sum_{j=4}^{m-1} \lambda(v_{j\ i}) \\
&= (1) + \left(4n + 4 + \frac{n+1}{2}\right) + \frac{(m-4)}{2}(3-n) + n\left(\frac{(m-1)\,m}{2} - 6\right) \\
&= \frac{3}{2}m + \frac{1}{2}n + \frac{1}{2}m^2 n - mn - \frac{1}{2} \\
\sum_{e\in E'} \lambda(e) &= \frac{m}{2}\left(2\,(m-1)\,n + 3 - n\right) + n\frac{m\,(m+1)}{2} \\
m\,(\lambda) &= \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 3m + \frac{1}{2}n + 2m^2 n - 2mn - \frac{1}{2}.
\end{aligned}$$

Case2: m is even and n is even:

$$\lambda\,(v_c) = n + 1 + \frac{n}{2},$$

$$\lambda\,(v_{1\ i}) = i,\ i = 1, 2, 3, ..., n,$$

$$\lambda(v_{2\ i}) = \begin{cases} 2n + 2 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ n + 1 + \frac{n}{2} - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v_{3\ i}) = \begin{cases} 2n + 2 + \frac{n}{2} - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ 3n + 2 - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v_{j\ i}) = \begin{cases} 1 + (j-1)\,n + i & , j = 4, 6, 8, ..., m-2,\ i = 1, 2, 3, ..., n \\ 2 + jn - i & , j = 5, 7, 9, ..., m-1,\ i = 1, 2, 3, ..., n \end{cases}$$

$$\lambda(e_{j\ i}) = \begin{cases} (m-1)\,n + 1 + (j-1)\,n + i & , j = 1, 3, 5, ..., m-1,\ i = 1, 2, 3, ..., n \\ (m-1)\,n + 2 + jn - i & , j = 2, 4, 6, ..., m,\ i = 1, 2, 3, ..., n \end{cases}$$

For all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, (m-1)\,n + 1\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_m, we have

$$\begin{aligned}
\sum_{v\in V'} \lambda(v) &= \lambda(v_c) + \lambda(v_{1\ i}) + \lambda(v_{2\ i}) + \lambda(v_{3\ i}) + \sum_{j=4}^{m-1} \lambda(v_{j\ i}) \\
&= \left(n + 1 + \frac{n}{2}\right) + \left(4n + 3 + \frac{n}{2}\right) + \frac{(m-4)}{2}(3-n) + n\left(\frac{(m-1)\,m}{2} - 6\right) \\
&= \frac{3}{2}m + 2n + \frac{1}{2}m^2 n - mn - 2 \\
\sum_{e\in E'} \lambda(e) &= \frac{m}{2}\left(2\,(m-1)\,n + 3 - n\right) + n\frac{m\,(m+1)}{2} \\
m\,(\lambda) &= \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 3m + 2n + 2m^2 n - 2mn - 2.
\end{aligned}$$

Case3: m is odd and n is odd:

$$\lambda(v_c) = 1,$$

$$\lambda(v_{j\ i}) = \begin{cases} 1 + (j-1)n + i & , j = 1,3,5,...,m-2,\ i = 1,2,3,...,n \\ 2 + jn - i & , j = 2,4,6,...,m-1,\ i = 1,2,3,...,n \end{cases}$$

$$\lambda(e_{1\ i}) = (m-1)n + 1 + i,\ i = 1,2,3,...,n$$

$$\lambda(e_{2\ i}) = \begin{cases} (m-1)n + n + \frac{n+1}{2} + i & , i = 1,2,3,...,\frac{n+1}{2} \\ (m-1)n + \frac{n+1}{2} + i & , i = \frac{n+1}{2} + 1, \ldots, n \end{cases}$$

$$\lambda(e_{3\ i}) = \begin{cases} (m-1)n + 3 + 3n - 2i & , i = 1,2,3,...,\frac{n+1}{2} \\ (m-1)n + 3 + 4n - 2i & , i = \frac{n+1}{2} + 1, \ldots, n \end{cases}$$

$$\lambda(e_{j\ i}) = \begin{cases} (m-1)n + 1 + (j-1)n + i & , j = 4,6,8,...,m-1,\ i = 1,2,3,...,n \\ (m-1)n + 2 + jn - i & , j = 5,7,9,...,m,\ i = 1,2,3,...,n \end{cases}$$

For all $v \in V$, we have $\lambda(v) \in \{1,2,3,\ldots,(m-1)n+1\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_m, we have

$$\begin{aligned} \sum_{v \in V'} \lambda(v) &= \lambda(v_c) + \sum_{j=2}^{m-1} \lambda(v_{j\ i}) \\ &= (1) + \frac{(m-1)}{2}(3-n) + n\frac{(m-1)m}{2} \\ &= \frac{3}{2}m + \frac{1}{2}n + \frac{1}{2}m^2 n - mn - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sum_{e \in E'} \lambda(e) &= \lambda(e_{1\ i}) + \lambda(e_{2\ i}) + \lambda(e_{3\ i}) + \sum_{j=4}^{m} \lambda(e_{j\ i}) \\ &= 4n + 3((m-1)n+1) + \frac{n+1}{2} + 1 + \frac{m-3}{2}(2(m-1)n + 3 - n) + n\left(\frac{m(m+1)}{2} - 6\right) \\ &= \frac{1}{2}m(3mn - 2n + 3) \end{aligned}$$

$$m(\lambda) = \sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e) = 3m + 2n + 2m^2 n - 2mn - 2.$$

Case4: m is odd and n is even:

$$\lambda(v_c) = n + 1 + \frac{n}{2},$$

$$\lambda(v_{1\ i}) = i,\ i = 1,2,3,...,n,$$

$$\lambda(v_{2\ i}) = \begin{cases} 2n + 2 - \frac{i+1}{2} & , i = 1,3,5,...,n-1 \\ n + 1 + \frac{n}{2} - \frac{i}{2} & , i = 2,4,6,\ldots,n \end{cases}$$

$$\lambda(v_{j\ i}) = \begin{cases} 1 + (j-1)n + i & , j = 3,5,7,...,m-2,\ i = 1,2,3,...,n \\ 2 + jn - i & , j = 4,6,8,...,m-1,\ i = 1,2,3,...,n \end{cases}$$

$$\lambda(e_{1\ i}) = \begin{cases} (m-1)\,n + 2 + \frac{n}{2} - \frac{i+1}{2} & ,i = 1,3,5,...,n-1 \\ (m-1)\,n + 2 + n - \frac{i}{2} & ,i = 2,4,6,\ldots,n \end{cases}$$

$$\lambda(e_{j\ i}) = \begin{cases} mn + 1 + (j-2)\,n + i & ,j = 2,4,6,...,m-1,\ i = 1,2,3,...,n \\ mn + 2 + (j-1)\,n - i & ,j = 3,5,7,...,m,\ i = 1,2,3,...,n \end{cases}$$

For all $v \in V$, we have $\lambda(v) \in \{1,2,3,\ldots,(m-1)\,n+1\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_m, we have

$$\sum_{v\in V'} \lambda(v) = \lambda(v_c) + \lambda(v_{1\ i}) + \lambda(v_{2\ i}) + \sum_{j=3}^{m-1} \lambda(v_{j\ i})$$

$$= \begin{cases} n + 1 + \frac{n}{2} + i + 2n + 2 - \frac{i+1}{2} + \frac{(m-3)}{2}(3-n) + n\left(\frac{(m-1)m}{2} - 3\right) & ,i = 1,3,5,...,n-1 \\ n + 1 + \frac{n}{2} + i + n + 1 + \frac{n}{2} - \frac{i}{2} + \frac{(m-3)}{2}(3-n) + n\left(\frac{(m-1)m}{2} - 3\right) & ,i = 2,4,6,\ldots,n \end{cases}$$

$$\sum_{e\in E'} \lambda(e) = \lambda(e_{1\ i}) + \sum_{j=2}^{m} \lambda(e_{j\ i})$$

$$= \begin{cases} (m-1)\,n + 2 + \frac{n}{2} - \frac{i+1}{2} + \frac{(m-1)}{2}(2mn + 3 - 3n) + n\left(\frac{(m+1)m}{2} - 1\right) & ,i = 1,3,5,...,n-1 \\ (m-1)\,n + 2 + n - \frac{i}{2} + \frac{(m-1)}{2}(2mn + 3 - 3n) + n\left(\frac{(m+1)m}{2} - 1\right) & ,i = 2,4,6,\ldots,n \end{cases}$$

$$m\,(\lambda) = \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 3m + 2n + 2m^2n - 2mn - 2.$$

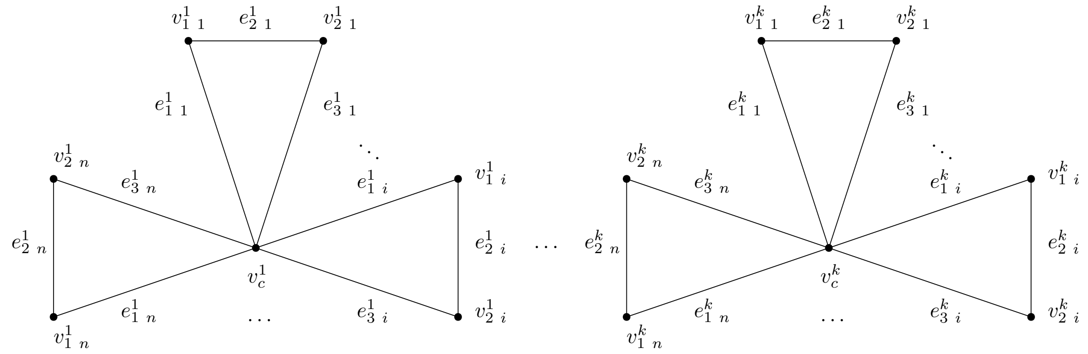
Hence F_n^m *admits a* C_m*-supermagic labeling.* □

Theorem 2.3. *Isomorphic copies of Friendship graph* $kF_n^3; n \geq 2, k \geq 2$, *admits a* C_3*-supermagic labeling.*

Proof. kF_n^3 has $k\,(2n+1)$ vertices and $3nk$ edges. The vertices and edges of kF_n^3 are denoted as follows:

$$V = \{v_c^s : s = 1,2,3,\ldots,k\} \cup \left\{v_{j\ i}^s : j = 1,2,\ i = 1,\ldots,n,\ s = 1,2,3,\ldots,k\right\}$$
$$E = \{e_{1\ i}^s : e_{1\ i}^s = v_c^s v_{1\ i}^s : i = 1,\ldots,n,\ s = 1,2,3,\ldots,k\}$$
$$\cup \{e_{2\ i}^s : e_{2\ i}^s = v_{1\ i}^s v_{2\ i}^s : i = 1,\ldots,n,\ s = 1,2,3,\ldots,k\}$$
$$\cup \{e_{3\ i}^s : e_{3\ i}^s = v_{2\ i}^s v_c^s : i = 1,\ldots,n,\ s = 1,2,3,\ldots,k\}$$

where v_c^s are the common verteces.



To define a bijection $\lambda : V \cup E \to \{1, 2, 3, \dots, |V| + |E|\}$, we need to investigate two cases.
Case1: n is odd:

$$\lambda\left(v_c^s\right) = (n+1)(k+1-s),$$
$$\lambda\left(v_{1\ i}^s\right) = (n+1)(s-1)+i,\ i = 1,2,3,\dots,n$$
$$\lambda(v_{2\ i}^s) = \begin{cases} (2k-1)n + k + \frac{n+1}{2} - 1 + i - (s-1)n & , i = 1,2,3,..., \frac{n+1}{2} \\ (2k-1)n + k + \frac{n+1}{2} - 1 + i - n - (s-1)n & , i = \frac{n+1}{2} + 1, \dots, n \end{cases}$$
$$\lambda\left(e_{1\ i}^s\right) = \begin{cases} (2k-1)n + k + 2n + 2 - 2i + (s-1)n & , i = 1,2,3,..., \frac{n+1}{2} \\ (2k-1)n + k + 2n + 2 - 2i + n + (s-1)n & , i = \frac{n+1}{2} + 1, \dots, n \end{cases}$$
$$\lambda(e_{2\ i}^s) = k + 3kn + (s-1)n + i,\ i = 1,2,3,,\dots,n$$
$$\lambda(e_{3\ i}^s) = k(2n+1) + 3kn + 1 - (s-1)n - i,\ i = 1,2,3,,\dots,n$$

where $s = 1,2,3,\dots,k$. Here, for all $v \in V$, we have $\lambda(v) \in \{1,2,3,\dots,k(2n+1)\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_3, we have

$$\sum_{v\in V'} \lambda(v) = \lambda(v_c^s) + \lambda(v_{1\ i}^s) + \lambda(v_{2\ i}^s)$$
$$= \begin{cases} 2k + \frac{1}{2}n + 3kn - ns - \frac{1}{2} + 2i & , i = 1,2,3,..., \frac{n+1}{2} \\ 2k - \frac{1}{2}n + 3kn - ns - \frac{1}{2} + 2i & , i = \frac{n+1}{2} + 1, \dots, n \end{cases}$$
$$\sum_{e\in E'} \lambda(e) = \lambda(e_{1\ i}^s) + \lambda(e_{2\ i}^s) + \lambda(e_{3\ i}^s)$$
$$= \begin{cases} 3k + 10kn + ns + 3 - 2i & , i = 1,2,3,..., \frac{n+1}{2} \\ 3k + n + 10kn + ns + 3 - 2i & , i = \frac{n+1}{2} + 1, \dots, n \end{cases}$$
$$m(\lambda) = \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 5k + \frac{1}{2}n + 13kn + \frac{5}{2}.$$

Case2: n is even:

$$\lambda\left(v_c^s\right) = kn + \frac{n}{2} + 1 + (n+1)(k-s),$$
$$\lambda\left(v_{1\ i}^s\right) = (s-1)n + i,\ i = 1,2,3,...,n,$$
$$\lambda(v_{2\ i}^s) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(s-1) + \frac{n}{2} + 1 - \frac{i+1}{2} & ,i = 1,3,5,...,n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(s-1) - \frac{i}{2} & ,i = 2,4,6,\dots,n \end{cases}$$
$$\lambda(e_{1\ i}^s) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)n + 1 - \frac{i+1}{2} & ,i = 1,3,5,...,n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)n + 1 + \frac{n}{2} - \frac{i}{2} & ,i = 2,4,6,\dots,n \end{cases}$$
$$\lambda(e_{2\ i}^s) = k + 3kn + (s-1)n + i,\ i = 1,2,3,...,n,$$
$$\lambda(e_{3\ i}^s) = k(2n+1) + 3kn + 1 - (s-1)n - i,\ i = 1,2,3,...,n,$$

where $s = 1,2,3,\dots,k$. Similarly, for all $v \in V$, we have $\lambda(v) \in \{1,2,3,\dots,k(2n+1)\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_3, we have

$$\sum_{v\in V'} \lambda(v) = \lambda(v_c^s) + \lambda(v_{1\ i}^s) + \lambda(v_{2\ i}^s)$$
$$= \begin{cases} k - \frac{1}{2}n + 3kn + ns + \frac{3}{2} + \frac{1}{2}i & ,i = 1,3,5,...,n-1 \\ k - n + 3kn + ns + 1 + \frac{1}{2}i & ,i = 2,4,6,\dots,n \end{cases}$$
$$\sum_{e\in E'} \lambda(e) = \lambda(e_{1\ i}^s) + \lambda(e_{2\ i}^s) + \lambda(e_{3\ i}^s)$$
$$= \begin{cases} 3k + \frac{1}{2}n + 11kn - ns + \frac{3}{2} - \frac{1}{2}i & ,i = 1,3,5,...,n-1 \\ 3k + n + 11kn - ns + 2 - \frac{1}{2}i & ,i = 2,4,6,\dots,n \end{cases}$$
$$m(\lambda) = \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = 4k + 14kn + 3.$$

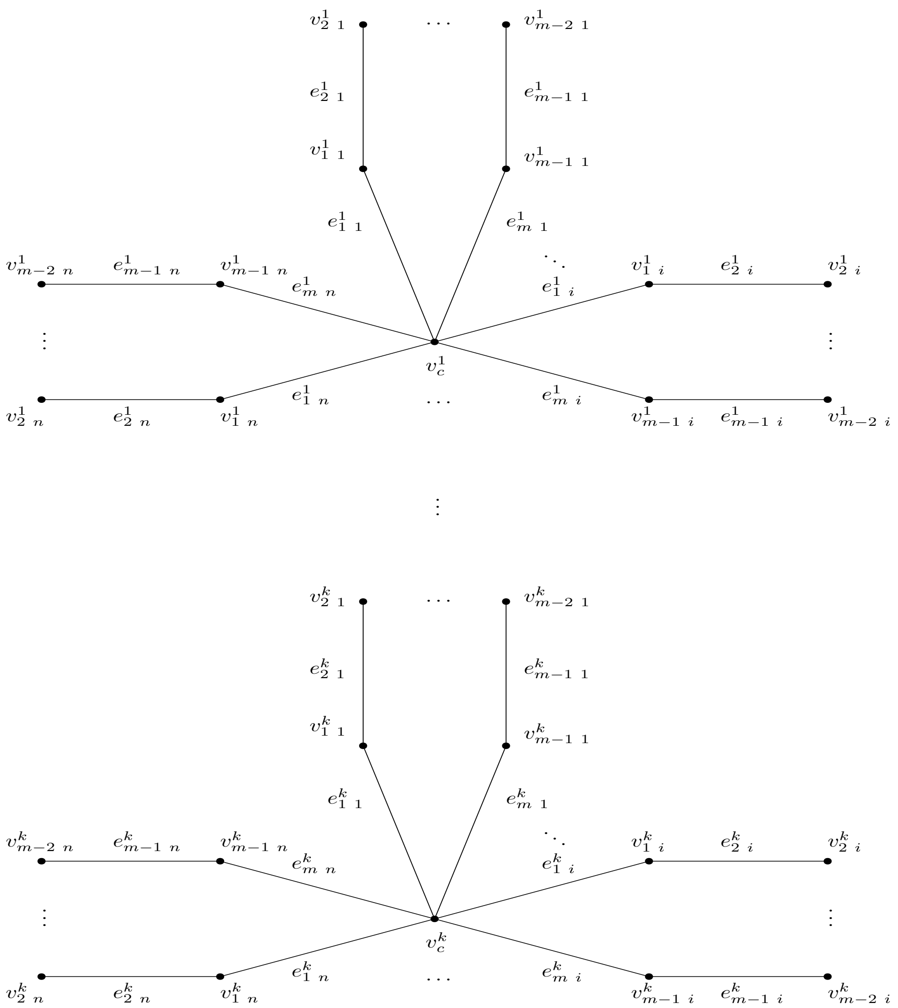
Hence kF_n^3 admits a C_3-supermagic labeling. □

Theorem 2.4. *k isomorphic copies of Friendship graph of C_m, $kF_n^m; m \geq 4, n \geq 2$, admits a C_m-supermagic labeling.*

Proof. kF_n^m has $k((m-1)n+1)$ vertices and kmn edges. The vertices and edges of kF_n^m are denoted as follows:

$$V = \{v_c^s : s = 1,2,3,\dots,k\} \cup \left\{v_{j\ i}^s : j = 1,2,3,\dots,m-1,\ i = 1,\dots,n,\ s = 1,2,3,\dots,k\right\}$$
$$E = \{e_{1\ i}^s : e_{1\ i}^s = v_c^s v_{1\ i}^s : i = 1,\dots,n,\ s = 1,2,3,\dots,k\}$$
$$\cup \left\{e_{j\ i}^s : e_{j\ i}^s = v_{j-1\ i}^s v_{j\ i}^s : j = 2,3,\dots,m-1,\ i = 1,\dots,n,\ s = 1,2,3,\dots,k\right\}$$
$$\cup \left\{e_{m\ i}^s : e_{m\ i}^s = v_{m-1\ i}^s v_c^s : i = 1,\dots,n,\ s = 1,2,3,\dots,k\right\}$$

where v_c^s are the common verteces.



To define a bijection $\lambda : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$, we need to investigate four cases.
Case1: n is even and m is even:

$$\lambda\left(v_c^s\right) = kn + \frac{n}{2} + 1 + (n+1)(k-s),$$

$$\lambda\left(v^s_{1\ i}\right) = (s-1)\,n + i,\ \ i = 1, 2, 3, ..., n,$$

$$\lambda(v^s_{2\ i}) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(s-1) + \frac{n}{2} + 1 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(s-1) - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v^s_{3\ i}) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)\,n + 1 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)\,n + 1 + \frac{n}{2} - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v^s_{j\ i}) = \begin{cases} k + (j-1)\,kn + (s-1)\,n + i & , j = 4, 6, 8, \ldots, m-2,\ i = 1, 2, 3, \ldots, n \\ k + jkn + 1 - (s-1)\,n - i & , j = 5, 7, 9, \ldots, m-1,\ i = 1, 2, 3, \ldots, n \end{cases}$$

$$\lambda(e^s_{j\ i}) = \begin{cases} k\left((m-1)\,n+1\right) + (j-1)\,kn + (s-1)\,n + i & , j = 1, 3, 5, , \ldots, m-1,\ i = 1, 2, 3, \ldots, n \\ k\left((m-1)\,n+1\right) + jkn + 1 - (s-1)\,n - i & , j = 2, 4, 6, \ldots, m,\ i = 1, 2, 3, \ldots, n \end{cases}$$

where $s = 1, 2, 3, \ldots, k$. Here, for all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, k(2n+1)\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_m, we have

$$\begin{aligned}
\sum_{v\in V'} \lambda(v) &= \lambda(v_c^s) + \lambda(v_{1\ i}^s) + \lambda(v_{2\ i}^s) + \lambda(v_{3\ i}^s) + \sum_{j=4}^{m-1} \lambda(v_{j\ i}^s) \\
&= \frac{1}{2}m - 2k + km + 2kn + \frac{1}{2}km^2n - kmn \\
\sum_{e\in E'} \lambda(e) &= \sum_{j=1}^{m} \lambda(e_{j\ i}^s) \\
&= \frac{1}{2}m\left(2k - 2kn + 3kmn + 1\right) \\
m(\lambda) &= \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = m - 2k + 2km + 2kn + 2km^2n - 2kmn.
\end{aligned}$$

Case2: n is even and m is odd:

$$\lambda\left(v_c^s\right) = kn + \frac{n}{2} + 1 + (n+1)(k-s),$$

$$\lambda\left(v_{1\ i}^s\right) = (s-1)n + i,\ i = 1, 2, 3, ..., n,$$

$$\lambda(v_{2\ i}^s) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(s-1) + \frac{n}{2} + 1 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(s-1) - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v_{3\ i}^s) = \begin{cases} kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)n + 1 - \frac{i+1}{2} & , i = 1, 3, 5, ..., n-1 \\ kn + \frac{n}{2} + 1 + (n+1)(k-1) + (k+1-s)n + 1 + \frac{n}{2} - \frac{i}{2} & , i = 2, 4, 6, \ldots, n \end{cases}$$

$$\lambda(v_{j\ i}^s) = \begin{cases} k + (j-1)kn + (s-1)n + i & , j = 4, 6, 8, \ldots, m-1,\ i = 1, 2, 3, \ldots, n \\ k + jkn + 1 - (s-1)n - i & , j = 5, 7, 9, \ldots, m-2,\ i = 1, 2, 3, \ldots, n \end{cases}$$

$$\lambda(e_{j\ i}^s) = \begin{cases} k((m-1)n+1) + (j-1)kn + (s-1)n + i & , j = 1, 3, 5, , \ldots, m-2,\ i = 1, 2, 3, \ldots, n \\ k((m-1)n+1) + jkn + 1 - (s-1)n - i & , j = 2, 4, 6, \ldots, m-1,\ i = 1, 2, 3, \ldots, n \end{cases}$$

$$\lambda(e_{m\ i}^s) = k((m-1)n+1) + mnk + 1 - (s-1)n - i,\ i = 1, 2, 3, \ldots, n$$

where $s = 1, 2, 3, \ldots, k$. Here, for all $v \in V$, we have $\lambda(v) \in \{1, 2, 3, \ldots, k(2n+1)\}$ and for any subgraph $H' = (V', E')$ isomorphic to C_m, we have

$$\begin{aligned}
\sum_{v\in V'} \lambda(v) &= \lambda(v_c^s) + \lambda(v_{1\ i}^s) + \lambda(v_{2\ i}^s) + \lambda(v_{3\ i}^s) + \sum_{j=4}^{m-1} \lambda(v_{j\ i}^s) \\
&= \frac{1}{2}m - 2k - n + km + \frac{3}{2}kn + ns + \frac{1}{2}km^2n - kmn - \frac{1}{2} + i \\
\sum_{e\in E'} \lambda(e) &= \sum_{j=1}^{m-1} \lambda(e_{j\ i}^s) + \lambda(e_{m\ i}^s) \\
&= \frac{1}{2}m + n + km + \frac{1}{2}kn - ns + \frac{3}{2}km^2n - kmn + \frac{1}{2} - i \\
m(\lambda) &= \sum_{v\in V'} \lambda(v) + \sum_{e\in E'} \lambda(e) = m - 2k + 2km + 2kn + 2km^2n - 2kmn.
\end{aligned}$$

Case3: n is odd and m is even:

$$\lambda\left(v_c^s\right)=(n+1)(k+1-s),$$
$$\lambda\left(v_{1\ i}^s\right)=(n+1)(s-1)+i,\ i=1,2,3,\ldots,n$$
$$\lambda(v_{2\ i}^s)=\begin{cases}(2k-1)n+k+\frac{n+1}{2}-1+i-(s-1)n & ,i=1,2,3,...,\frac{n+1}{2}\\ (2k-1)n+k+\frac{n+1}{2}-1+i-n-(s-1)n & ,i=\frac{n+1}{2}+1,\ldots,n\end{cases}$$
$$\lambda\left(v_{3\ i}^s\right)=\begin{cases}(2k-1)n+k+2n+2-2i+(s-1)n & ,i=1,2,3,...,\frac{n+1}{2}\\ (2k-1)n+k+2n+2-2i+n+(s-1)n & ,i=\frac{n+1}{2}+1,\ldots,n\end{cases}$$
$$\lambda(v_{j\ i}^s)=\begin{cases}k+(j-1)kn+(s-1)n+i & ,j=4,6,8,\ldots,m-2,\ i=1,2,3,\ldots,n\\ k+jkn+1-(s-1)n-i & ,j=5,7,9,\ldots,m-1,\ i=1,2,3,\ldots,n\end{cases}$$
$$\lambda(e_{j\ i}^s)=\begin{cases}k((m-1)n+1)+(j-1)kn+(s-1)n+i & ,j=1,3,5,,\ldots,m-1,\ i=1,2,3,\ldots,n\\ k((m-1)n+1)+jkn+1-(s-1)n-i & ,j=2,4,6,\ldots,m,\ i=1,2,3,\ldots,n\end{cases}$$

where $s=1,2,3,\ldots,k$. Here, for all $v\in V$, we have $\lambda(v)\in\{1,2,3,\ldots,k(2n+1)\}$ and for any subgraph $H'=(V',E')$ isomorphic to C_m, we have

$$\begin{aligned}\sum_{v\in V'}\lambda(v)&=\lambda(v_c^s)+\lambda(v_{1\ i}^s)+\lambda(v_{2\ i}^s)+\lambda(v_{3\ i}^s)+\sum_{j=4}^{m-1}\lambda(v_{j\ i}^s)\\ &=\frac{1}{2}m-k+\frac{1}{2}n+km+kn+\frac{1}{2}km^2n-kmn-\frac{1}{2}\\ \sum_{e\in E'}\lambda(e)&=\sum_{j=1}^{m}\lambda(e_{j\ i}^s)\\ &=\frac{1}{2}m(2k-2kn+3kmn+1)\\ m(\lambda)&=\sum_{v\in V'}\lambda(v)+\sum_{e\in E'}\lambda(e)=m-k+\frac{1}{2}n+2km+kn+2km^2n-2kmn-\frac{1}{2}.\end{aligned}$$

Case4: n is odd and m is odd:

$\lambda\left(v_c^s\right)=(n+1)(k+1-s)$,

$\lambda\left(v_{1\ i}^s\right)=(n+1)(s-1)+i,\ i=1,2,3,\ldots,n$

$$\lambda(v_{2\ i}^s)=\begin{cases}(2k-1)n+k+\frac{n+1}{2}-1+i-(s-1)n & ,i=1,2,3,...,\frac{n+1}{2}\\ (2k-1)n+k+\frac{n+1}{2}-1+i-n-(s-1)n & ,i=\frac{n+1}{2}+1,\ldots,n\end{cases}$$

$$\lambda(v_{3\ i}^s)=\begin{cases}(2k-1)n+k+2n+2-2i+(s-1)n & ,i=1,2,3,...,\frac{n+1}{2}\\ (2k-1)n+k+2n+2-2i+n+(s-1)n & ,i=\frac{n+1}{2}+1,\ldots,n\end{cases}$$

$$\lambda(v_{j\ i}^s)=\begin{cases}k+(j-1)kn+(s-1)n+i & ,j=4,6,8,\ldots,m-1,\ i=1,2,3,\ldots,n\\ k+jkn+1-(s-1)n-i & ,j=5,7,9,\ldots,m-2,\ i=1,2,3,\ldots,n\end{cases}$$

$$\lambda(e_{j\ i}^s)=\begin{cases}k((m-1)n+1)+(j-1)kn+(s-1)n+i & ,j=1,3,5,,\ldots,m-2,\ i=1,2,3,\ldots,n\\ k((m-1)n+1)+jkn+1-(s-1)n-i & ,j=2,4,6,\ldots,m-1,\ i=1,2,3,\ldots,n\end{cases}$$

$\lambda(e_{m\ i}^s)=k((m-1)n+1)+mnk+1-(s-1)n-i,\ i=1,2,3,\ldots,n$

where $s=1,2,3,\ldots,k$. Here, for all $v\in V$, we have $\lambda(v)\in\{1,2,3,\ldots,k(2n+1)\}$ and for any subgraph $H'=(V',E')$ isomorphic to C_m, we have

$$\begin{aligned}\sum_{v\in V'}\lambda(v)&=\lambda(v_c^s)+\lambda(v_{1\ i}^s)+\lambda(v_{2\ i}^s)+\lambda(v_{3\ i}^s)+\sum_{j=4}^{m-1}\lambda(v_{j\ i}^s)\\&=\frac{1}{2}m-k-\frac{1}{2}n+km+\frac{1}{2}kn+ns+\frac{1}{2}km^2n-kmn-1+i\\ \sum_{e\in E'}\lambda(e)&=\sum_{j=1}^{m-1}\lambda(e_{j\ i}^s)+\lambda(e_{m\ i}^s)\\&=\frac{1}{2}m+n+km+\frac{1}{2}kn-ns+\frac{3}{2}km^2n-kmn+\frac{1}{2}-i\\ m(\lambda)&=\sum_{v\in V'}\lambda(v)+\sum_{e\in E'}\lambda(e)=m-k+\frac{1}{2}n+2km+kn+2km^2n-2kmn-\frac{1}{2}.\end{aligned}$$

Hence kF_n^m admits a C_m-supermagic labeling. □

3. Conclusions

In this paper, we gave class of C_m-supermagic labeling of friendship graphs and isomorphic copies of friendship graphs.

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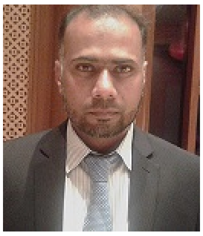

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