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New Numerical Approach for Solving Abel's Integral Equations

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Abstract: In this article, we present an efficient method for solving Abel's integral equations. This important equation is consisting of an integral equation that is modeling many problems in literature. Our proposed method is based on first taking the truncated Taylor expansions of the solution function and fractional derivatives, then substituting their matrix forms into the equation. The main character behind this technique's approach is that it reduces such problems to solving a system of algebraic equations, thus greatly simplifying the problem. Numerical examples are used to illustrate the preciseness and effectiveness of the proposed method. Figures and tables are demonstrated to solutions impress. Also, all numerical examples are solved with the aid of Maple.

Keywords: Integral equations, singular integral equations, generalized Taylor series, approximate solutions, collocation method.

1. Introduction

Abel's integral equations provide an essential tool for modeling various phenomena in basic and engineering sciences such as physics, chemistry, biology, electronics, mechanics, and analyzing laser-induced breakdown spectroscopy [5,13,18,2,27]. Cimatti G. considers Abel's integral equation to solve an inverse problem in thermoelectricity[4]. Abel's problem is as follows: Find a curve in the vertical XoY so that a material point which has started its motion at a point of the curve with ordinate x without initial velocity and moving along the curve under the action of gravity without friction, will reach the axis X in time $t = f(x)/\sqrt{2g}$ where g is the acceleration in free-falling.

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Abel's equation is an integral equation derived directly from a concrete problem of mechanics or physics. In chronological order, Abel's problem is the first to lead to the study of integral equations.

Abel's integral equations often appear in two forms; the first and second kind as follows respectively:

$$f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \ , \qquad (1)$$

and

$$y(x) = f(x) - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \ , \qquad (2)$$

where $f(x)$ is a continuous function in $[0,1]$. Also, we can write generalized Abel's integral equation in the following forms:

$$f(x) = \int_0^x \frac{y(t)}{(x-t)^\beta} dt \ , \qquad (3)$$

and

$$y(x) = f(x) - \int_0^x \frac{y(t)}{(x-t)^\beta} dt \ , \qquad (4)$$

where $1 > \beta > 0, f(x) \in [0,1], 0 \leq x, t \leq 1$.[2]

There are too many applications on Abel's equation. Brenke W.C. [2] gives an application of Abel's equation. He considers a flow of a stream.

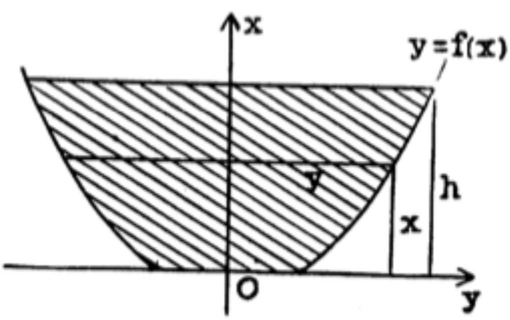

Figure 1 The cross-section of a barrage nick [2]

Let the shaded region in Figure 1 demonstrate the cross-section of a barrage nick, and the crosscut is symmetrical with respect to the x-axis. Nick flow amount per unit time is given

$$Q = C \int_0^h \sqrt{h-x}\, f(x) dx, \qquad (5)$$

where the form of nick is determined by $y = f(x); x \geq 0$. Here the aim is to find $f(x)$ thereby the amount of flow per unit time will be proportional to a given depth of flow power so $Q = k'h^m, m > 0$. As a result, the main idea here is to find $f(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h-x} f(x)dx = kh^m. \quad (6)$$

Let differentiate both sides of the given integral equation with respect to h gives

$$\int_0^h \frac{f(x)}{\sqrt{h-x}} dx = 2kmh^{m-1}. \quad (7)$$

The last integral equation is a form of Abel's integral equation.
In the last two decades, many effective and simple methods have been proposed and applied successfully to various singular integral equations with a wide range of applications [14,7,15,17,1,26,23,20].
In this paper, we use fractional calculus properties to solve these singular integral equations. There are many fractional derivative definitions in the literature, like Caput-Fabrizio derivative [3], M-fractional derivative [16], etc. In this study, we use Caputo fractional derivative definition.
Fractional calculus is used to model real-world problems. Many researchers are studying this subject [22,25,11,9,28,24,10,8,12].
Fractional calculus can reduce computations and improve solutions. Since the calculation of fractional integral and derivative are directly challenging for arbitrary functions and fractional differential equation. Because of this reason, we can calculate the approximate solution of functions by using the generalized Taylor series [19].
In this study, we seek the approximate solution of Eq. (1) with the fractional Taylor series as $D_a^{k\alpha} y(x) \in C(a,b]$:

$$y_N(x) = \sum_{k=0}^{N} \frac{(x-c)^{k\alpha}}{\Gamma(k\alpha+1)} (D_a^{k\alpha} y)(c), \quad (8)$$

where $0 < \alpha \leq 1$. We use the generalized Taylor matrix method. This method transforms each part of the equation into matrix form; then, we get the linear algebraic equation. Then this equation is solved. We obtain the generalized Taylor coefficients, the approximate solutions for various N. All computations are performed on the computer algebraic system Maple 13.

2. Basic Definitions

In this section, we provide some basic definitions and then present the properties of fractional calculus [6,21].

Definition 2.1 The Riemann-Liouville fractional derivative of order α to the variable t and with the starting point at $t = a$ is

$$ {}_aD^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha+m+1)} \frac{d^{m+1}}{dt^{m+1}} \int_a^t (t-\tau)^{m-\alpha} f(\tau) d\tau, 0 \le m \le \alpha < m+1, \\ \frac{d^m f(t)}{dt^m}, \alpha = m+1 \in \mathbb{N}, \end{cases} \tag{9} $$

such that $D^0 f(x) = f(x)$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f(x) \in C_\mu$, $\mu \ge -1$, is defined as

$$ J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \ \alpha > 0, \ x > 0, \tag{10} $$

such that $J^0 f(x) = f(x)$.

Definition 2.3 The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$$ D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, $$

for n -1<α< n, n ∈N $t > 0$, $f \in C_{-1}^n$.

Definition 2.4 Riemann –Liouville fractional derivative ${}_aD_t^{\alpha} f(t)$ of the power function $f(t) = (t-a)^v$, where v is a real number is

$$ D^{\alpha} \text{ (t-a)} v = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)}) (t-a)^{\ v-\alpha}, $$

Some properties of the fractional derivative and fractional integral are listed below [6,21]:

1. $D^{\alpha}(\lambda f(t) + \mu f(t)) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} f(t)$, λ, μ are constants,
2. D^{α} $((J^{\alpha} f(t)) = f(t)$,
3. $D^{\alpha}(D^{\beta} f(t)) = D^{\alpha+\beta} f(t)$,
4. $D^{\alpha} C = 0$ for any constant C.

Theorem 1. (Generalized Taylor Formula) Suppose that $D_a^{k\alpha} f(x) \in C(a, b]$ for $k = 0, 1, \dots, n+1$ where $0 < \alpha \le 1$ then we have [19]

$$ f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha} f\right)(a) + \frac{\left(D^{(n+1)} f\right)(\xi)}{\Gamma((n+1)\alpha+1)} \left(x - a \ \ \right)^{(n+1)\alpha} \ , $$

with $a \le \xi \le x$, $\forall x \in (a, b]$, where

$$ D_a^{n\alpha} = D_a^{\alpha}.D_a^{\alpha}.D_a^{\alpha}.\dots D_a^{\alpha} \quad (n \text{ times}). $$

3. Description of the method

In this method, we use the generalized Taylor series through the fractional calculus to reach the approximate solution of Abel's integral equations. The first and the second kind of Abel's integral equation should be considered at this point. According to Equations (3), (4), and (9), Abel's integral equations of the first and second kind can be written as follows:

$$f(x) = \Gamma(1-\beta)J^{1-\beta}y(x), \tag{11}$$

and

$$y(x) = f(x) - \Gamma(1-\beta)J^{1-\beta}y(x). \tag{12}$$

We apply the operator $D^{1-\beta}$ on both sides of Eq. (11) and (12), we obtain the fractional differential equation

$$D^{\alpha}f(x) = \Gamma(\alpha)y(x), \tag{13}$$

and

$$D^{\alpha}y(x) = D^{\alpha}f(x) - \Gamma(\alpha)y(x), \tag{14}$$

where $\alpha = 1-\beta \in (0,1)$.
Since calculating $D^{\alpha}y(x)$ is directly cost and inefficient, we use generalized Taylor series for approximating$y(x)$. We first consider the solution $y(x)$ of Eq. (13) and Eq. (14) defined by a truncated generalized Taylor series (8). Then, we have the matrix form of the solution $y_N(x)$:

$$[y_N(x)] = XM_0A\ , \tag{15}$$

where

$$X = [1 \quad (x-c)^{\alpha} \quad (x-c)^{2\alpha} \quad \cdots \quad (x-c)^{N\alpha}],$$

$$M_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)} \end{bmatrix}, A = \begin{bmatrix} D_*^{0\alpha}y(c) \\ D_*^{1\alpha}y(c) \\ D_*^{2\alpha}y(c) \\ \vdots \\ D_*^{N\alpha}y(c) \end{bmatrix},$$

where $D^0y_N(x) = y_N(x)$.
Now, we consider the differential part of $D^{1\alpha}y(x)$ Eq. (13) and Eq. (14) $i = n, n-1, \dots, 0$.
For $i = 1$, we obtained the matrix representation of the function $D_*^{1\alpha}y_N(x)$

$$D_*^{1\alpha}y_N(x) = D_*^{1\alpha}XM_0A, \tag{16}$$

and we compute the $D_*^{1\alpha}X$, then

$$D_*^{1\alpha}X = [D_*^{\alpha}1 \quad D_*^{\alpha}(x-c)^{\alpha} \quad D_*^{\alpha}(x-c)^{2\alpha} \quad \cdots \quad D_*^{\alpha}(x-c)^{N\alpha}]$$
$$= \left[0 \quad \frac{\Gamma(\alpha+1)}{\Gamma(1)} \quad \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} \quad \cdots \quad \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{(N-1)\alpha}\right]$$
$$= XM_1,$$

where

$$M_1 = \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Afterward, the matrix representation of $D_*^{1\alpha}y_N(x)$ as

$$D_*^{\alpha}y_N(x) = D_*^{1\alpha}y_N(x) = \mathrm{XM}_1 M_0 A \,. \tag{17}$$

Substituting Eq. (15) and Eq. (17) into Eq. (13) and Eq. (14), we obtain the fundamental matrix relation of the Eq. (13), and Eq. (14) are

$$\Gamma(\alpha)X(x)M_1M_0A = g(x), \tag{18}$$

and

$$(X(x)M_1M_0 + \Gamma(\alpha)X(x)M_1M_0)A = g(x), \tag{19}$$

where $g(x) = D^{\alpha}f(x)$.

4. Method of solution

The collocation points are defined by

$$x_i = \frac{i}{N}, \; i = 0,1,\dots,N,$$

we can write the Eq. (18) and Eq (19)

$$\Gamma(\alpha)X(x_i)M_1M_0A = g(x_i), \tag{20}$$

and

$$(X(x_i)M_1M_0 + \Gamma(\alpha)X(x_i)M_1M_0)A = g(x_i), \tag{21}$$

in short, the fundamental matrix equation

$$\mathrm{BXM}_1 M_0 A = F \tag{22}$$

and

$$(XM_1M_0 + BXM_1M_0)A = F, \tag{23}$$

where

$$X=\begin{bmatrix}1 & (x_0-c)^{\alpha} & (x_0-c)^{2\alpha} & \cdots & (x_0-c)^{N\alpha}\\ 1 & (x_1-c)^{\alpha} & (x_1-c)^{2\alpha} & \cdots & (x_1-c)^{N\alpha}\\ 1 & (x_2-c)^{\alpha} & (x_2-c)^{2\alpha} & \cdots & (x_2-c)^{N\alpha}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & (x_N-c)^{\alpha} & (x_N-c)^{2\alpha} & \cdots & (x_N-c)^{N\alpha}\end{bmatrix},\ F=\begin{bmatrix}g(x_0)\\ g(x_1)\\ g(x_2)\\ \vdots\\ g(x_N)\end{bmatrix}$$

$$B=\begin{bmatrix}\Gamma(1-\alpha) & 0 & 0 & \cdots & 0\\ 0 & \Gamma(1-\alpha) & 0 & \cdots & 0\\ 0 & 0 & \Gamma(1-\alpha) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \Gamma(1-\alpha)\end{bmatrix}.$$

Hence, the fundamental matrix equation (22) and (23) corresponding to Eq. (13) and Eq. (14) can be written in the form of

$$\mathrm{WA}=F \text{ or } [\mathrm{W;F}],\ W=[w]_{i,j},\ i,j=0,1,\ldots,N. \tag{24}$$

If rank$W=N+1$we obtain the coefficient matrix A:

$$A=W^{-1}F. \tag{25}$$

We can quickly check the accuracy of the method. Since the truncated Taylor series (3) is an approximate solution of Eq. (1), when the solution $y_N(x)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x=x_q\in,[0,1]$, $q=0,1,2,\ldots\infty$.

$$E_{1N}(x_q)=\left|y(x_q)+\int_0^{x_q}\frac{y(t)}{(x_q-t)^{\beta}}dt-f(x_q)\right|\cong 0,$$

$$E_{2N}(x_q)=\left|\int_0^{x_q}\frac{y(t)}{(x_q-t)^{\beta}}dt-f(x_q)\right|\cong 0.$$

5. Examples

To illustrate the effectiveness of the method we proposed in this paper, several numerical examples are carried out in this section. In the following computations, absolute errors between the-order approximate values and the corresponding exact values and maximum error are determined for convenience. All of the calculations have been performed using Maple 13.

Example 1: Abel's integral equation of the second kind is considered as follows [1]:

$$y(x)=x+\frac{4}{3}x^{\frac{3}{2}}-\int_0^x\frac{y(t)}{\sqrt{x-t}}dt. \tag{26}$$

Now, we can apply our technique described in Section 4 in Eq. (26) for $N = 6$, $c = 0$, that is:

$$y_6(x) = \sum_{k=0}^{6} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} (D_*^{k\alpha} y(x))_{x=0},$$

with collocation points

$$x_0 = 0, x_1 = 1/5, x_2 = 2/5, x_3 = 3/5, x_4 = 4/5, x_5 = 1.$$

Firstly, Eq. (26) transform a fractional differential equation with Eq. (14)

$$D^{\frac{1}{2}} y(x) = D^{\frac{1}{2}}(x + \tfrac{4}{3} x^{\frac{3}{2}}) - \Gamma(\tfrac{1}{2}) y(x), \tag{27}$$

and

$$D^{\frac{1}{2}} y(x) + \Gamma(\tfrac{1}{2}) y(x) = \frac{\Gamma(2)}{\Gamma(1.5)} \sqrt{x} + \frac{4\Gamma(2.5)}{3\Gamma(2)} x \quad . \tag{28}$$

Then, the fundamental matrix relation of this equation is

$$(\mathrm{X}\mathrm{M}_1 M_0 + \mathrm{B}\mathrm{X}\mathrm{M}_0) A = F, \tag{29}$$

where the matrices are

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ 0.859117 \\ 1.422631 \\ 1.937511 \\ 2.427216 \\ 2.900833 \end{bmatrix}, \ X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \sqrt{5}/5 & 1/5 & \sqrt{5}/25 & 1/25 & \sqrt{5}/125 \\ 1 & \sqrt{5}\sqrt{2}/5 & 2/5 & 2\sqrt{5}\sqrt{2}/25 & 4/25 & 4\sqrt{5}\sqrt{2}/125 \\ 1 & \sqrt{5}\sqrt{3}/5 & 3/5 & 3\sqrt{5}\sqrt{3}/25 & 9/25 & 9\sqrt{5}\sqrt{3}/125 \\ 1 & \sqrt{5}\sqrt{4}/5 & 4/5 & 4\sqrt{5}\sqrt{4}/25 & 16/25 & 16\sqrt{5}\sqrt{4}/25 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then, we have

$$[W\,;F] =$$

$$\begin{bmatrix} 1.772453 & 1 & 0 & 0 & 0 & 0 \ ; & 0 \\ 1.772453 & 1.894442 & 0.859117 & 0.319256 & 0.102732 & 0.029540; & 0.859117 \\ 1.772453 & 2.264911 & 1.422631 & 0.737309 & 0.332102 & 0.133969; & 1.422631 \\ 1.772453 & 2.549193 & 1.937511 & 1.219677 & 0.668657 & 0.328722; & 1.937511 \\ 1.772453 & 2.788854 & 2.427216 & 1.754055 & 1.105453 & 0.625297; & 2.427216 \\ 1.772453 & 3 & 2.900833 & 2.333333 & 1.638479 & 1.033333; & 2.900833 \end{bmatrix}.$$

and solving this equation, we obtained the coefficients of the generalized Taylor series

$$A = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T.$$

Hence, for $N = 5$, the approximate solution of Example 1 is

$$y_5 = x,$$

which is the exact solution to this problem. Since the exact solution's degree of the polynomial is 1, this method gives the exact solution for $N \geq 2$.

Example 2: The first type, Abel's integral equation [1,26], is considered as

$$\int_0^x \frac{y(t)}{\sqrt{x-t}} dt = \frac{2}{105}(105 - 56x^2 + 48x^3), \tag{30}$$

with the exact solution $y(x) = x^3 - x^2 + 1$. We applied the generalized Taylor series with the collocation method and solved Eq. (30). We have approximate solutions for $N = 7$:

$y_7(x) = 1 - 0.758E - 11\sqrt{x} + 0.725E - 10x - 0.282E - 9x^{\frac{3}{2}} - x^2 - 0.641E - 9x^{\frac{5}{2}} + x^3 - 0.902E - 10x^{\frac{7}{2}}$;
for $N = 8$:

$$y_8(x) = 1 - 0.352E - 16\sqrt{x} + 0.288E - 15x - 0.940E - 13x^{\frac{3}{2}} - x^2 - 0.141E - 13x^{\frac{5}{2}} + x^3 - 0.281E$$
$$-13x^{\frac{7}{2}} + 0.568E - 14x^4.$$

We say that approximate solutions and exact solutions are the same.

Example 3: We consider Abel equation [15] as

$$\int_0^x \frac{y(t)}{(x-t)^{1/3}} dt = f(x) \tag{31}$$

with $f(x) = x^{\frac{5}{3}}$ and its exact solution is $y(x) = \frac{10}{9}x$. Eq. (31) turn into a fractional differential equation with Eq. (14)

$$\Gamma(\tfrac{2}{3})y(x) = \frac{\Gamma(8/3)}{\Gamma(2)}x. \tag{32}$$

We seek the approximate solutions y_N by Taylor series, for $c = 0$, $N = 4$:

$$y_4(x) = \sum_{k=0}^{4} \frac{x^{k\alpha}}{\Gamma(\alpha k + 1)} (D_*^{k\alpha} y(x))_{x=0}$$

with collocation points are

$$x_0 = 0, x_1 = 1/4, x_2 = 2/4, x_3 = 3/4, x_4 = 1.$$

Then, the fundamental matrix relation of this equation is

$$BXM_0A = F, \tag{33}$$

where

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\frac{\sqrt{3}}{\pi}\Gamma(\frac{2}{3}) & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2}\frac{\sqrt{3}}{\pi}\Gamma(\frac{2}{3})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{8}\frac{\sqrt{3}}{\pi}\Gamma(\frac{2}{3}) \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0.376143 \\ 0.752287 \\ 1.128431 \\ 1.504575 \end{bmatrix},$$

$$B = \begin{bmatrix} \Gamma(\frac{2}{3}) & 0 & 0 & 0 & 0 \\ 0 & \Gamma(\frac{2}{3}) & 0 & 0 & 0 \\ 0 & 0 & \Gamma(\frac{2}{3}) & 0 & 0 \\ 0 & 0 & 0 & \Gamma(\frac{2}{3}) & 0 \\ 0 & 0 & 0 & 0 & \Gamma(\frac{2}{3}) \end{bmatrix}.$$

Substituting these matrices in Eq. (33), we have an algebraic linear equation system with five unknowns:

$$[W;F] = \left[\begin{array}{ccccc|c} 1.35411793 & 0 & 0 & 0 & 0 & 0 \\ 1.35411793 & 0.95527482 & 0.59517539 & 0.33852948 & 0.17911402 & 0.37614387 \\ 1.35411793 & 1.20357086 & 0.94494078 & 0.67705896 & 0.45133907 & 0.75228774 \\ 1.35411793 & 1.37774470 & 1.23822718 & 1.01558845 & 0.77498139 & 1.12843161 \\ 1.35411793 & 1.51640426 & 1.5 & 1.3541179 & 1.13730319 & 1.50457548 \end{array}\right],$$

and so we solve the linear algebraic equation with five unknowns and obtain the coefficients of the Taylor series

$$A = [0 \quad 0.442E-7 \quad -0.171E-6 \quad 1.111111 \quad -0.115E-6].$$

Hence, for $N = 4$, the approximate solution of Example 3 is as follows:

$$y_4(x) = 0.495E - 7x^{\frac{1}{3}} - 0.189E - 6x^{\frac{2}{3}} + 1.1111111x - 0.972E - 7x^{\frac{4}{3}}.$$

We compare the absolute errors, E_N and e_N for $N = 4\ and\ 6$ in Table 1.

Table 1. The numerical results of Example 3

x	$N=4$	E_4	$N=6$	E_6
0.0	0.000E-0	0.000E-0	0.000E-0	0.000E-0
0.2	0.309E-9	0.700E-9	0.125E-14	0.282E-13
0.4	0.303E-10	0.500E-10	0.392E-15	0.240E-13
0.6	0.286E-9	0.400E-9	0.842E-16	0.240E-13
0.8	0.542E-9	0.600E-9	0.684E-15	0.220E-13
e_N	0.238E-6		0.203E-14	

Example 4: We consider Abel equation [1,15]

$$\int_0^x \frac{y(t)}{(x-t)^{1/2}}dt = e^x - 1.$$

The exact solution to this problem is $y(x) = e^x erf(\sqrt{x})/\sqrt{\pi}$. A comparison between the exact solution and the generalized Taylor series solutions is given in Table 2. Table 3 shows a comparison $E_N(x)$ for some N. Moreover, in table 4, we compare the present method with several numerical methods. Figure 2 and Figure 3 display the approximate solutions-exact solution and absolute errors for various N, respectively. In Figure 4, we focus on the absolute error for $N_e = 9$. Also, we display comparisons of some numerical results in Figure 5.

Table 2. The numerical result for Example 4

x	Exact Solution	*N=4*	*Ne=4*	*N=7*	*Ne=7*	*N=9*	*Ne=9*
0.0	0.000000	0.000000	0.000E-0	0.000000	0.000E-0	0.000000	0.000E-0
0.2	0.325884	0.324725	0.115E-2	0.325880	0.362E-5	0.325884	0.255E-7
0.4	0.529333	0.529823	0.489E-3	0.529333	0.216E-6	0.529333	0.204E-8
0.6	0.747040	0.746743	0.296E-3	0.747040	0.938E-7	0.747040	0.709E-9
0.8	0.997089	0.997306	0.216E-3	0.997089	0.228E-6	0.997089	0.781E-9
e_N		10^{-2}		10^{-5}		10^{-7}	

Table 3. Comparison of $E_N(x)$ for some N for Example 4

x	E_4	E_7	E_9
0.0	0.000E-0	0.000E-0	0.000E-0
0.2	0.422E-2	0.447E-4	0.239E-5
0.4	0.200E-2	0.305E-4	0.165E-5
0.6	0.181E-2	0.244E-4	0.133E-5
0.8	0.150E-2	0.209E-4	0.115E-5

Table 4. The comparison of some numerical methods

x	Exact solution	Huang[15] $N = 3$	Avazzadeh[1] $N = 20$	Present met. ($N = 4$)
0.1	0.21529	0.21629	0.21520	0.20881
0.2	0.32588	0.37727	0.32593	0.32472
0.3	0.42756	0.42925	0.42779	0.42808
0.4	0.52933	0.53126	0.52927	0.52982
0.5	0.63503	0.63715	0.63491	0.63503
0.6	0.74704	0.74933	0.74719	0.74674
0.7	0.86718	0.86962	0.86718	0.86700
0.8	0.99708	0.99963	0.99692	0.99730
0.9	1.13829	1.14091	1.13760	1.13879

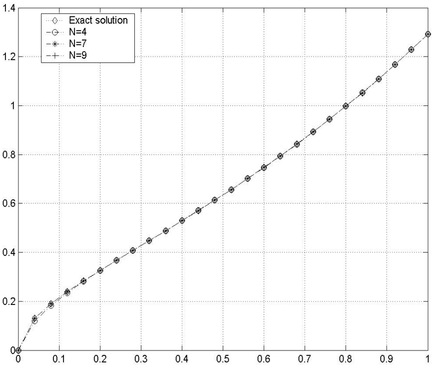

Figure 2. The comparison of exact and approximate solution

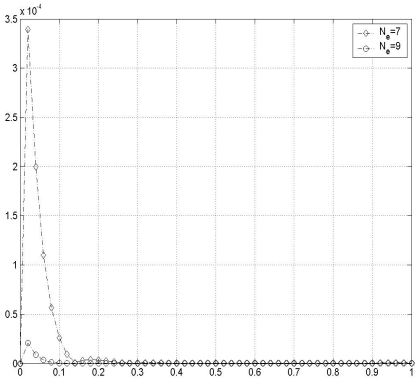

Figure 3. The comparison of the error function

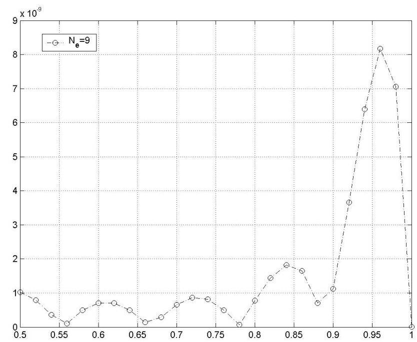

Figure 4. The absolute errors for $N = 9$

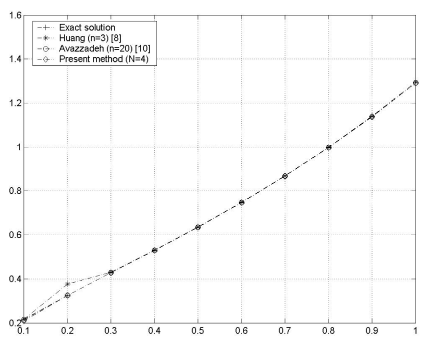

Figure 5 The comparisons of some numerical methods

Example 5: Considering the Abel integral equation of the first kind as follows [19]

$$x = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt.$$

The exact solution to this problem is $y(x) = \frac{2}{\pi}\sqrt{x}$. We approximately solve this problem for various *N,* and we obtain the exact solution for $N \geq 1$.

Example 6: Considering the Abel integral equation of the second kind as follows [1]:

$$y(x) = x^2 + \frac{16}{15} x^{\frac{5}{2}} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt.$$

The exact solution is $y(x) = x^2$. Numerical results are given in Table 5, and absolute errors are displayed in Figure 6.

Table 5. The numerical results for Example 6

x	$N = 4$	E_4	$N = 5$	E_5
0.0	0.940E-12	0.940E-12	0.549E-13	0.549E-13
0.2	0.458E-12	0.932E-12	0.269E-13	0.584E-13
0.4	0.371E-12	0.935E-12	0.217E-13	0.548E-13
0.6	0.320E-12	0.934E-12	0.188E-13	0.548E-13
0.8	0.285E-12	0.929E-12	0.169E-13	0.548E-13
1.0	0.255E-12	0.915E-12	0.154E-13	0.548E-13

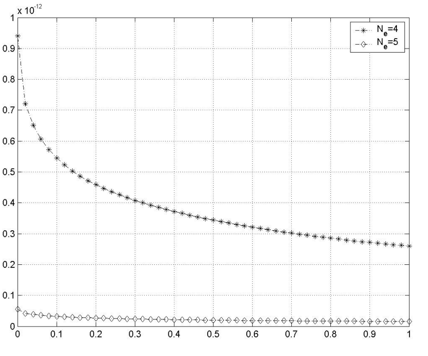

Figure 6. The comparison of absolute errors for $N = 4,5$.

6. Conclusion

Abel's integral equation is fundamental in literature. The present study aims to develop an efficient and accurate method for solving singular Volterra integral equations. The problem has been reduced to solving a system of linear algebraic equations. We note that this method is easy for computation and running. The numerical examples demonstrate that the accuracy of lower-order approximations is very satisfactory. We have demonstrated the accuracy and efficiency of the present method. The convergence of our method can be seen in Figures2-6. Tables 1-2 and 5 showed that the error decreased as N increased. In Table 3, we determined the comparison of errors for different N values. In table 4, we compared the method we presented with the values found by different methods. The examples show that the Taylor collocation method has been successfully applied to finding the approximate solutions Abel integral equation. Also, the method can be expanded to solve a fractional system of Abel's integral equation.

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